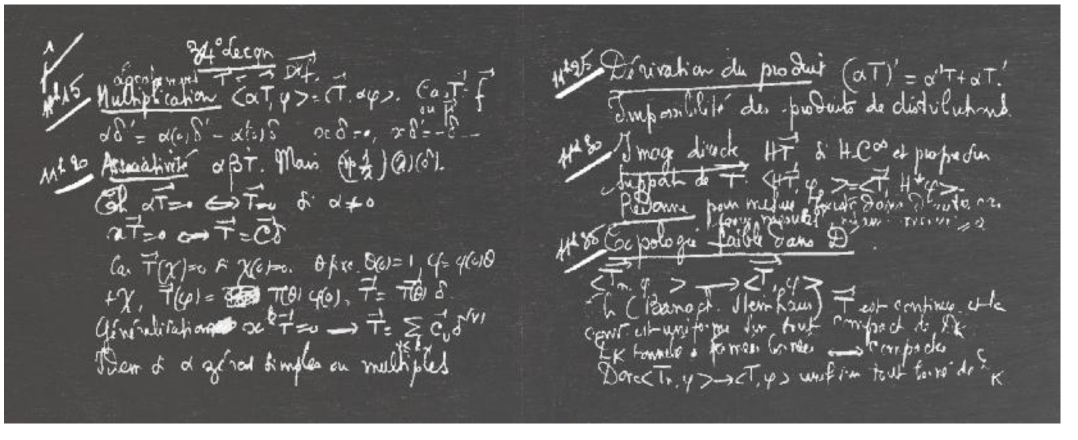
# *Theory of Distributions*

*Giovanni* Di Fratta
work in progress

TECHNISCHE
UNIVERSITÄT
WIEN

[101.873] Theory of distributions

[2020W] Lecture Notes



Préparation d'un amphi: notes manuscrites de Laurent Schwartz
*Ecole polytechnique, Fonds L. Schwartz*

«The invention of distributions occurred in Paris, in early November 1944. The discovery was quite sudden, taking place in a single night. I always called the night of my discovery a marvelous night, the most beautiful night of my life. On this particular night, I felt sure of myself and filled with a sense of exaltation. I lost no time in rushing to explain everything in detail to Cartan, who as I mentioned earlier, lived next door. He was enthusiastic: "*There you are, you've just resolved all the difficulties of differentiation*! *Now, we'll never again have functions without derivatives*" he told me. If a function has no (Weierstrass) derivative, then this simply means that its derivatives are operators, but not functions».

**Laurent Schwartz**

## Table of contents.

# 1

## Basic algebraic concepts

### 1.1 | Minkowski operations in vector spaces

Let $X$ be a vector space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$). Let $A$ and $B$ be subsets of $X$, $\Lambda$ a subset of $\mathbb{K}$. We set

$$A+B := \{x\in X :: x=a+b \text{ for some } (a,b)\in A\times B\} \quad (1.1)$$
$$\Lambda A := \{x\in X :: x=\lambda a \text{ for some } (\lambda,a)\in \mathbb{K}\times A\}. \quad (1.2)$$

The concept is named after Hermann Minkowski (1864–1909) who was a German mathematician and professor at Königsberg, Zürich and Göttingen.

We say that $A+B$ is the **(Minkowski) sum** of the sets $A$ and $B$, and $\Lambda A$ is the **(Minkowski) product** of the set of scalars $\Lambda$ and the set of vectors $A$. Note that, if $A, B$ are vector subspaces of $X$ then $A+B$ is a vector subspace of $X$ as well; besides, $\Lambda A = A$ for every $\Lambda\subseteq\mathbb{K}$.

**Notation.** When $A$ is a singleton set, say $A=\{a\}$, we write $a+B$ instead of $\{a\}+B$ and $\Lambda a$ instead of $\Lambda\{a\}$. Similarly, if $\Lambda$ is a singleton set, say $\Lambda=\{\lambda\}$, we write $\lambda A$ instead of $\{\lambda\}A$. Finally, we set $-A:=(-1)A$ and $A-B:=A+(-B)$. We say that $A-B$ is the **algebraic difference** of the sets $A$ and $B$. With this notation, we have

$$A+B = \cup_{a\in A}(a+B), \quad (1.3)$$
$$\Lambda A = \cup_{\lambda\in\Lambda}\lambda A. \quad (1.4)$$

It is evident that if $A, B, C$ are subsets of $X$ and $\lambda\in\mathbb{K}$, then

$$\begin{gathered} A+B=B+A, \quad (A+B)+C=A+(B+C), \\ \lambda(A+B)=\lambda A+\lambda B. \end{gathered} \quad (1.5)$$

In other words, the Minkowski sum is commutative and associative and, when $\Lambda:=\{\lambda\}\subseteq\mathbb{K}$ is a singleton, the product is left-distributive over the sum. Clearly, $\emptyset+A=A+\emptyset=\emptyset A=\Lambda\emptyset=\emptyset$. ⋯

**1.1. Remark.** The multiplication fails to be left-distributive over the sum when $\Lambda\subseteq\mathbb{K}$ is *not* a singleton: in general, one has $\Lambda(A+B)\subseteq\Lambda A+\Lambda B$. Also, note that the sum is not left-distributive over the product; indeed, in general, one can only guarantee the inclusion

$$(\lambda+\mu)A\subseteq\lambda A+\mu A. \quad (1.6)$$

The equality in the previous relation does not hold even when $\lambda=\mu$; thus, e.g., $2A\subseteq A+A$ (cf. Figure 1.1). The notion of convex set, introduced later on, guarantees the left-distributivity of the sum over the product, i.e., the equality in the previous relation when $\lambda,\mu\geqslant 0$. ⋯

**1.2. Remark.** The algebraic difference $A-B:=\{x\in X :: x=a-b, (a,b)\in A\times B\}$ must not be confused with the so-called **Minkowski difference** (or **geometric difference**), usually denoted by the same symbol $A-B$, but defined as the set $\{c\in X :: c+B\subseteq A\}$. For example, if $A=[-2,2]\subseteq\mathbb{R}$ and $B=[-1,1]\subseteq\mathbb{R}$ then the Minkowski difference of $A$ and $B$ is $[-1,1]$ whereas $A-B=A+B=[-3,3]$.

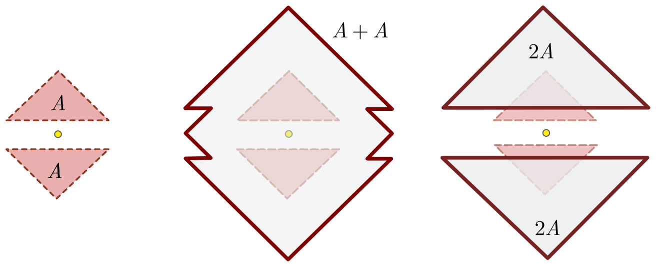



**Figure 1.1.** The sum is not left-distributive over the product; indeed, in general, one can only guarantee the inclusion $(\lambda+\mu)A \subseteq \lambda A + \mu A$. Thus, in general, $2A \subseteq A + A$.

Also, $A - B$ must not be confused with the set-theoretic difference $A \setminus B := A \cap B^{\complement}$ which is the relative complement of $B$ in $A$, i.e., the set that contains exactly those elements belonging to $A$ but not to $B$. The set-theoretic difference $A \setminus B$ is often denoted by $A - B$, but here we shall avoid this use. ⋯

▶ Let us point out some immediate (but useful) consequences of the definition, which will be often used in the sequel. If $X$ and $Y$ are vector spaces over the same field $\mathbb{K}$ and $f: X \to Y$ is a **linear** map, then, for every $\Lambda \subseteq \mathbb{K}$, $A, B \subseteq X$, and $E, F \subseteq Y$ we have:

$$f(\Lambda A) = \Lambda f(A), \qquad f(A+B) = f(A) + f(B), \tag{1.7}$$

$$\Lambda f^{-1}(E) \subseteq f^{-1}(\Lambda E), \qquad f^{-1}(E) + f^{-1}(F) \subseteq f^{-1}(E+F). \tag{1.8}$$

Note that, in general, relations (1.8) do not hold with the equality sign. Indeed, if the relation $\Lambda f^{-1}(E) \subseteq f^{-1}(\Lambda E)$ in (1.8) holds with an equality sign for every $\Lambda \subseteq \mathbb{K}$ then, for $\Lambda = \{0\}$, one gets $\{0\} = \ker f$, i.e., that $f$ is injective. Also, if $f$ is *not* surjective and $f^{-1}(E) + f^{-1}(F) = f^{-1}(E+F)$ for every $E, F \subseteq Y$, then there exists $\emptyset \neq E_0 \subseteq Y$ such that $f^{-1}(E_0) = \emptyset$; this implies (set $E := E_0$ and $F := Y$) $\emptyset = f^{-1}(E_0) + f^{-1}(Y) = f^{-1}(E_0 + Y) = f^{-1}(Y)$. But this cannot be the case as $\{0\} \subseteq f^{-1}(Y)$. ⋯

○

▶ Also, if $A \subseteq B$ then $\lambda A \subseteq \lambda B$. In particular, $-A \subseteq -B$. In general, it is not true that if $|\lambda| \leqslant |\mu|$ then $|\lambda| A \subseteq |\mu| B$; however, this is true for the so-called *balanced sets* defined later (cf. **Proposition 1.24**). ⋯

▶ Moreover, $\lambda(A \cup B) = \lambda A \cup \lambda B$ and $\lambda(A \cap B) = \lambda A \cap \lambda B$. The proof of all these assertions is straightforward. For example, one has $\Lambda f(A) = \bigcup_{(\lambda,x)\in\Lambda\times A}\{\lambda f(x)\} = \bigcup_{(\lambda,x)\in\Lambda\times A}\{f(\lambda x)\} = f(\Lambda A)$. ⋯

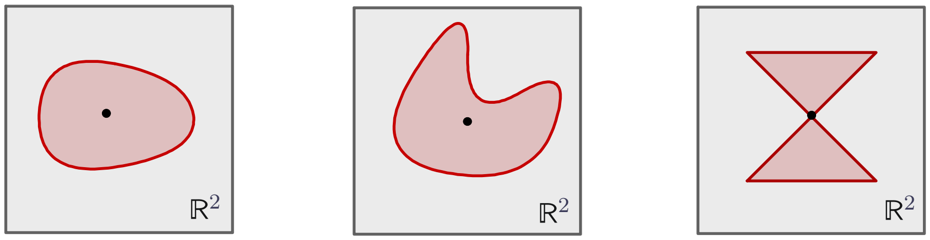
## 1.2 Absorbing sets, balanced sets, convex sets

Let $X$ be a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Let $A$ and $B$ subsets of $X$.

**1.3. Definition.** We say that $A$ is a **symmetric set** (with respect to the origin) if $A \subseteq -A$. In other words, $A$ is symmetric if, and only if, $x \in A \Rightarrow -x \in A$. Equivalently, $A$ is symmetric if, and only if, $A = -A$ (if $A \subseteq -A$ the equality holds by multiplying both sides of the inclusion by $-1$). ⋞

○

**1.4. Definition.** In Euclidean geometry, a **line segment** is a part of a line that is bounded by two distinct endpoints, and contains every point on the line between its endpoints. For any $x \in X$ we




Absorbing, convex, not balanced | Absorbing, not convex, not balanced | Not absorbing, not convex, balanced

**Figure 1.2.** Some geometric examples in $\mathbb{R}^2$ showing the notions of absorbing set, convex set, and balanced set. **Note** that a necessary condition for a set $A$ to be balanced is that it is symmetric: $-A \subseteq A$.

denote by $[-x,x]_{\mathbb{K}}$ the **symmetric** (with respect to the origin) **segment** ending at $x$; that is

$$[-x,x]_{\mathbb{K}} := \{\lambda x :: \lambda \in \mathbb{D}_\bullet\} = \mathbb{D}_\bullet x = \bigcup_{\rho \in \mathbb{D}_\bullet} \{\rho x\}, \tag{1.9}$$

with $\mathbb{D}_\bullet$ the **closed** unit disk of the complex plane if $\mathbb{K} = \mathbb{C}$ and $\mathbb{D}_\bullet = [-1,1]$ if $\mathbb{K} = \mathbb{R}$.

Note that, for every $x \in X$, the set $[-x,x]_{\mathbb{K}}$ is a symmetric set. Also note that for every $\sigma \in \mathbb{K}$ we have

$$\sigma \cdot [-x,x]_{\mathbb{K}} \equiv [-\sigma x, \sigma x]_{\mathbb{K}}. \tag{1.10}$$

**Example 1.5.** Consider $X = \mathbb{C}$ as a vector space over $\mathbb{R}$; for $x := (1,0)$ we have $[-x,x]_{\mathbb{R}} = [-1,1] \times \{0\}$. However, if $X = \mathbb{C}$ is considered as a vector space over $\mathbb{C}$, then $[-x,x]_{\mathbb{C}} = \mathbb{D}_\bullet$. Eventually, note the limiting case $[-0,0]_{\mathbb{R}} = [-0,0]_{\mathbb{C}} = \{0\}$.

**1.6. Definition.** We say that $A$ is **balanced** (or **circled**, or **équilibrée**) if it is nonempty and $\rho A \subseteq A$ for every $\rho \in \mathbb{K}$ such that $|\rho| \leqslant 1$. When $\mathbb{K} = \mathbb{C}$ this is equivalent to say that $\mathbb{D}_\bullet A \subseteq A$. In other words, the set $A$ is balanced whenever $\rho v \in A$ for every $v \in A$ and every $\rho \in \mathbb{D}_\bullet$ (every $\rho \in [-1,1]_{\mathbb{R}}$ in the real case).

**1.7. Remark.** The condition that a balanced set has to be nonempty is not so relevant. In fact, in literature, very often, this condition is not included and, in this case, the emptyset is balanced. However, to avoid continually writing that a property holds for a balanced set *provided that it is nonempty*, we require that a balanced set is nonempty by the very definition.

**1.8. Remark.** Note that, in the definition of balanced set, it is possible to replace the inclusion sign with the equality sign; that is, $A$ is balanced if, and only if, $\mathbb{D}_\bullet A = A$ (because $1 \in \mathbb{D}_\bullet$). Similarly, $A$ is symmetric if, and only if, $A = -A$ (just multiply by $-1$ both sides of the inclusion $A \subseteq -A$).

○

Note that a necessary condition for a set $A$ to be balanced is that $-A = A$, i.e., that $A$ is symmetric (because of $-1 \in \mathbb{D}_\bullet$). Moreover, every balanced set $A$ must pass through the origin, i.e., $0 \in A$ if $A$ is balanced, because of $0 \in \mathbb{D}_\bullet$ (note that, if one allows the emptyset to be balanced, then one has to specify that a balanced set pass through the origin *provided that it is nonempty*). Actually, when $\mathbb{K} = \mathbb{R}$ and $A$ is a balanced set, given any point $v \in A$, the set $A$ must contain the whole (symmetric) *real* segment $[-v,v]_{\mathbb{R}}$. Similarly, when $\mathbb{K} = \mathbb{C}$ and $A$ is a balanced set, given any point $v \in A$, the set $A$ contains the (symmetric) *complex* segment $[-v,v]_{\mathbb{C}}$. In fact, $A$ is balanced if, and only if, $A = \bigcup_{v \in A} (\bigcup_{|\rho| \leqslant 1} \lambda v)$. Summarizing, the following geometric characterization holds.

**1.9. Proposition.** *Let $X$ be a vector space over $\mathbb{K}$. A set $A \subseteq X$ is balanced if, and only if,*

$$\forall v \in A \quad [-v, v]_{\mathbb{K}} \subseteq A. \tag{1.11}$$

*In particualr, every balanced set is symmetric and passes through the origin.*

**1.10. Remark.** The underlying field $\mathbb{K}$ plays an important role in the definition of balanced set. For example, the symmetric segment $[-1,1]_{\mathbb{R}}$, which is nothing but the closed interval $[-1,1] \times \{0\} \subseteq \mathbb{C}$, is a balanced subset of $\mathbb{C}$ considered as a vector space over $\mathbb{K} = \mathbb{R}$. However, the same closed interval $[-1,1] \times \{0\} \subseteq \mathbb{C}$ is no more a balanced subset of $\mathbb{C}$ if $\mathbb{C}$ is considered as a vector space over $\mathbb{K} = \mathbb{C}$. Indeed, for $\rho := i \in \mathbb{D}_\bullet$ we get $\rho \cdot [-1,1]_{\mathbb{R}} = [-i, i]_{\mathbb{R}} \nsubseteq [-1,1]_{\mathbb{R}}$.

○

The next concept will be essential to define, later on, the notion of bounded subset in topological vector spaces.

**1.11. Definition.** Let $X$ be a vector space, $A, B \subseteq X$ two sets. We say that $A$ **absorbs** $B$ (or that $B$ is **absorbed by** $A$) if there exists a $\lambda_0 > 0$ ($\lambda_0 \neq \infty$) such that $\lambda A \supseteq B$ for every $\lambda \in \mathbb{K}$ such that $|\lambda| \geqslant \lambda_0$. This can be expressed as

$$B \subseteq \bigcap\nolimits_{|\lambda| \geqslant \lambda_0} \lambda A. \tag{1.12}$$

In other words, $A$ absorbs $B$ if there exists a $\lambda_0 > 0$ such that for every $b \in B$ one has $b \in \lambda A$ for every $|\lambda| \geqslant \lambda_0$. Note that, $\lambda_0$ depends both on $A$ and $B$.

**1.12. Remark. ($A$ absorbs $B$, dual formulation)** Since $b \in \lambda A$ with $\lambda > 0$ if, and only if, $\rho b \in A$ with $\rho := \lambda^{-1}$, we get that $A$ absorbs $B$ if there exists a $\rho_0 > 0$ sufficiently small such that $\rho b \in A$ for every $|\rho| \leqslant \rho_0$ and every $b \in B$; that is, if

$$\bigcup\nolimits_{|\rho| \leqslant \rho_0} \rho B \subseteq A. \tag{1.13}$$

We stress that, in particular, $\rho_0$ must be different from zero — otherwise every set would absorb any other set.

**1.13. Definition.** Let $X$ be a vector space. We say that $A$ is **absorbing** (or **absorbent**, or **radial at the origin**) if it absorbs all singletons of the space. In symbols, $A$ is absorbing if for every $x \in X$ there exists a $\lambda_0(x) > 0$ ($\lambda_0(x) \neq \infty$) such that $x \in \lambda A$ for every $\lambda \in \mathbb{K}$ such that $|\lambda| \geqslant \lambda_0(x)$; that is

$$x \in \bigcap\nolimits_{|\lambda| \geqslant \lambda_0(x)} \lambda A. \tag{1.14}$$

Note that for $\lambda \neq 0$ we have $x \in \lambda A \Leftrightarrow \lambda^{-1} x \in A$, therefore if $\mathbb{K} = \mathbb{C}$ then $A$ is absorbing if, and only if, for every $x \in X$ there exists a sufficiently small disk $\mathbb{D}_{\bullet,\rho}$ (closed disk centered at $0 \in \mathbb{C}$ and of radius $\rho > 0$) such that $\mathbb{D}_{\bullet,\rho} \subseteq A$. **Note that** *any absorbing set must contain the null vector $0 \in X$.*

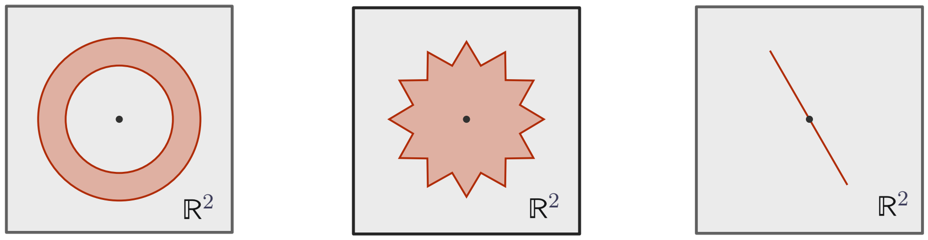
**1.14. Remark. ($A$ is absorbing, dual formulation)** Since $x \in \lambda A$ if, and only if, $\rho x \in A$ with $\rho := \lambda^{-1}$, we get that $A$ is absorbing if there exists a $(0 \neq)\, \rho_0(x) > 0$ sufficiently small such that $\rho x \in A$ for every $|\rho| \leqslant \rho_0(x)$; that is, if

$$\bigcup\nolimits_{|\rho| \leqslant \rho_0(x)} \{\rho x\} \subseteq A. \tag{1.15}$$

**1.15. Proposition.** *Let $X$ be a vector space over $\mathbb{K}$. A set $A \subseteq X$ is absorbing if, and only if,*

$$\forall x \in X\ \exists \rho_0(x) > 0 \ :: \ \rho_0(x) \cdot [-x, x]_{\mathbb{K}} \subseteq A.$$

*In other words, $A$ is absorbing if, and only if, it absorbs every symmetric segment in $X$.*




Not absorbing, not balanced, not convex | Absorbing, balanced, not convex | Not absorbing, balanced, convex

**Figure 1.3.** Some geometric examples in $\mathbb{R}^2$ showing the notions of absorbing set, convex set, and balanced set. **Note** that a necessary condition for a set $A$ to be balanced is that $-A \subseteq A$.

**1.18. Definition.** We say that $A$ is **convex** if $\lambda A + \mu A \subseteq A$ for every (nonnegative) real numbers $\lambda$, $\mu \geqslant 0$ such that $\lambda + \mu = 1$. Note that, in the definition of convex set, it is possible to replace the inclusion sign $\subseteq$ with the equality sign $=$. Indeed, the reverse inclusion $A = (\lambda + \mu)A \subseteq \lambda A + \mu A$ always holds, regardless of whether $A$ is convex or not (cf. (1.6)).

Note that a generic element of $\lambda A + \mu A$ has the form $\lambda a_1 + \mu a_2$ with $a_1, a_2 \in A$ and, in general, $a_1 \neq a_2$.

**1.19. Remark.** Note that the emptyset is convex. Here we are not excluding this eventuality as it creates no exposition issues.

**1.20. Remark.** Note that, in considering a sum among sets we can always collect terms which have a common factor, i.e., it is always true that $\lambda A + \lambda B = \lambda(A + B)$. However, in general, we cannot collect terms which are scaled by different factors as in $\lambda A + \mu A$ to obtain $(\lambda + \mu)A$. The notion of convexity allows for that equality at least when $\lambda, \mu \geqslant 0$. (cf. **Proposition 1.24**).

**Example 1.21.** ▶ The singleton $\{0\}$ of a vector space, is a balanced set because $[-0, 0]_{\mathbb{K}} = \{0\}$. It is never absorbing unless the vector space is trivial, i.e., $X = \{0\}$ — indeed, if $X \neq \{0\}$ and $0 \neq x \in X$, then for every $\sigma > 0$ one has $\sigma \cdot [-x, x]_{\mathbb{K}} \nsubseteq \{0\}$. Finally, $\{0\}$ is convex because any singleton is convex. If $X \neq \{0\}$ and $0 \neq x \in X$ then $\{x\}$ is convex, but it is neither balanced nor absorbing because $\{x\}$ does not pass through the origin.

▶ In a normed space, both the open and closed balls centered at the origin are absorbing, convex and balanced.


Convex, balanced and absorbing | Convex, balanced and absorbing

**Figure 1.4.** Some geometric examples in $\mathbb{R}^2$ showing the notions of absorbing set, convex set, and balanced set. **Note** that a necessary condition for a set $A$ to be balanced is that $-A \subseteq A$.

In $\mathbb{R}^2$ the *unit* ball $B(c)$ centered at the point $c := (1/2, 0)$ is absorbing but *not balanced*.

○

**Example 1.22.** ▶ Every vector subspace of $X$ is balanced but *not* necessarily absorbing. For example, consider the vector space $C(I)$, of all continuous and *real-valued* functions defined on the compact set $I = [0,1] \subseteq \mathbb{R}$. The set $\mathbb{R}[x]$ of all polynomial functions defined on $I$ is a vector subspace of $C(I)$ and therefore balanced. Nevertheless, it is not absorbing. Indeed, the definition of absorbing set specializes to $\mathbb{R}[x]$ in $C(I)$ as: *for every continuous function $u \in C(I)$ there exists a $\sigma_0(u) > 0$ such that $\sigma u \in \mathbb{R}[x]$ for every $|\sigma| \leqslant \sigma_0(u)$*. But this is clearly false because it is not the case that every continuous function is a scalar multiple of a polynomial. It is clear how to generalize what we learned from $\mathbb{R}[x]$:

**1.23. Proposition.** *Proper subspaces of a vector space are never absorbing.*

**Exercise 1.5.** Let $X$ be a vector space over $\mathbb{K}$ and $M \trianglelefteq X$ a *proper* vector subspace of $X$, i.e., there exists $x \neq 0$ such that $x \in X \setminus M$. Prove that $M$ is not absorbing.

**Solution.** If $0 \neq x \in X \setminus M$ then for every $|\sigma| > 0$ we have $\sigma x \notin M$ because $M$ is a vector space (if $\sigma x \in M$ then also $\sigma^{-1}(\sigma x) = x \in M$, a contradiction).

**1.24. Proposition.** *The following properties analyze the behavior of the notions introduced so far with respect to the algebraic operations on $X$.*

***i.*** *If $A$ is a* **balanced set** *then for any $\lambda \in \mathbb{K}$ the rescaled set $\lambda A$ is balanced too and*

$$\lambda A = |\lambda| A.$$

*Moreover, if $|\lambda| \leqslant |\mu|$ then $\lambda A = |\lambda| A \subseteq |\mu| A = \mu A$. Note that, in general, this is not true. Think about an annulus in $\mathbb{R}^2$.*

***ii.*** *Let $A$ be a balanced set and $B$ any subset of $X$. To check if $A$ absorbs $B$ it is then sufficient to check if there exists a $\lambda_0 \in \mathbb{K}$ such that $\lambda_0 A \supseteq B$.* **In particular:** *a sufficient condition for a balanced set $A$ to be absorbing is the existence, for every $x \in X$, of a $\rho_0(x) \neq 0$ such that $\rho_0(x) x \in A$.*

***iii.*** *Let $A$ be a subset of $X$. Then, $A$ is convex if, and only if,* **for every** $\lambda, \mu$ **positive or null** *one has*

$$(\lambda + \mu) A = \lambda A + \mu A.$$

○

**Proof.** ***i.*** Assume that $A$ is balanced. Then $\mathbb{D}_\bullet(\lambda A) = \lambda(\mathbb{D}_\bullet A) = \lambda A$. This proves that $\lambda A$ is balanced.

Next, recall the definition of balanced set. By assumption, $A = \mathbb{D}_\bullet A$. Thus, the assertion $\lambda A = |\lambda| A$ follows from the fact that $\lambda \mathbb{D}_\bullet = |\lambda| \mathbb{D}_\bullet$ for every $\lambda \in \mathbb{K}$. Precisely, since $A$ and $\lambda A$ are balanced, we have

$$\lambda A = \mathbb{D}_\bullet(\lambda A) = (\lambda \mathbb{D}_\bullet) A = (|\lambda| \mathbb{D}_\bullet) A = |\lambda| (\mathbb{D}_\bullet A) = |\lambda| A.$$

**Moreover**, if $|\lambda| \leqslant |\mu| \neq 0$ then $|\lambda| / |\mu| \in \mathbb{D}_\bullet$ so that $\frac{|\lambda|}{|\mu|} A \subseteq \mathbb{D}_\bullet A = A$, because $A$ is balanced. That is, $|\lambda| A \subseteq |\mu| A$. It follows that

$$\lambda A = |\lambda| A \subseteq |\mu| A = \mu A.$$

***ii.*** Suppose that $\lambda_0 A \supseteq B$ for some $\lambda_0 \in \mathbb{K}$. Since $A$ is balanced, by statement ***i.***, we have that $\lambda A \supseteq \lambda_0 A$ for every $|\lambda| \geqslant |\lambda_0|$. Hence, $\lambda A \supseteq B$ for every $|\lambda| \geqslant |\lambda_0|$. But this, by definition, means that $A$ absorbs $B$.

***iii.*** If $(\lambda + \mu)A = \lambda A + \mu A$ for every non-negative $\lambda, \mu \in \mathbb{R}$ then in particular, $\lambda A + (1 - \lambda)A = A$ for every $0 \leqslant \lambda \leqslant 1$ and, therefore, $A$ is convex. Conversely, assume that $A$ is convex. If $\{\lambda, \mu\} = \{0\}$, then the relation $(\lambda + \mu)A = \lambda A + \mu A$ is clearly satisfied. On the other hand, if $\{\lambda, \mu\} \neq \{0\}$, we set $\alpha := \frac{\lambda}{\lambda + \mu}$ and $\beta := \frac{\mu}{\lambda + \mu}$. Then, we have (recall that, in general there holds $\alpha A + \alpha B = \alpha(A + B)$)

$$\lambda A + \mu A = (\lambda + \mu)(\alpha A) + (\lambda + \mu)(\beta A) = (\lambda + \mu)(\alpha A + \beta A) = (\lambda + \mu)A.$$

The last equality follows from the convexity of $A$ because of $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$. ■■■■

**1.25. Proposition.** *Let $X$ be a vector space over $\mathbb{K}$. The following assertions hold:*

- ***i.*** *The intersection of a* **finite** *number of absorbing sets in $X$ is still an absorbing set in $X$.*
- ***ii.*** *The intersection of a family (no cardinality constraint) of balanced sets in $X$ is still a balanced set in $X$.*
- ***iii.*** *The intersection of a family (no cardinality constraint) of convex sets in $X$ is still a convex set in $X$.*

*Note that, in general, property* ***i.*** *does not hold for an arbitrary family of absorbing sets. Just think about the family $(\overline{B(1/n)})_{n \in \mathbb{N}}$ of closed balls of a normed space centered at the origin and of radius $1/n$.*

**Proof.** ***i.*** Let $A_1, A_2$ be teo absorbing sets. By definition, for every $x \in X$ there exist $\rho_1, \rho_2 > 0$ such that

$$\begin{aligned} \rho x \in A_1 \quad \forall \rho \leqslant \rho_1, \\ \rho x \in A_2 \quad \forall \rho \leqslant \rho_2. \end{aligned}$$

Setting $\rho_0 := \rho_1 \wedge \rho_2$ we have that $\rho x \in A_1 \cap A_2$ for every $\rho \leqslant \rho_0$.

***ii.*** First, observe that for any family of (generic) subsets $(A_j)_{j \in \Theta}$ of $X$ and any subset $\Lambda \subseteq \mathbb{K}$ we have $\Lambda(\cap_{j \in \Theta} A_j) \subseteq \cap_{j \in \Theta}(\Lambda A_j)$. Now, if $(A_j)_{j \in \Theta}$ is a family of balanced sets. We have

$$\mathbb{D}_\bullet(\cap_{j \in \Theta} A_j) \;\subseteq\; \cap_{j \in \Theta}(\mathbb{D}_\bullet A_j) \;=\; \cap_{j \in \Theta} A_j.$$

***iii.*** First, observe that for any pair of families $(A_j, B_j)_{j \in \Theta}$ made up of (generic) subsets of $X$ and any $\lambda \in \mathbb{K}$ we have

$$(\cap_{j \in \Theta} A_j) + (\cap_{j \in \Theta} B_j) \subseteq \cap_{j \in \Theta}(A_j + B_j).$$

Now, if $(A_j)_{j \in \Theta}$ is a family of convex sets, for every $\lambda, \mu \geqslant 0$ such that $\lambda + \mu = 1$ we have

$$\begin{aligned} \lambda(\cap_{j \in \Theta} A_j) + \mu(\cap_{j \in \Theta} A_j) \;&\subseteq\; \cap_{j \in \Theta}(\lambda A_j) + \cap_{j \in \Theta}(\mu A_j) \\ &\subseteq\; \cap_{j \in \Theta}(\lambda A_j + \mu A_j) \end{aligned}$$

$$= \bigcap_{j\in\Theta} A_j.$$

The proof is complete. ■

**1.26. Proposition.** *Let $X$ be a vector space over $\mathbb{K}$. The following assertions hold:*

*i.* *The union of a family (no cardinality constraint) of absorbing sets is absorbing. Actually, it is sufficient that just one set of the family is absorbing for the unions to be absorbing (because to be absorbing is a property preserved by supersets).*

*ii.* *The union of a family (no cardinality constraint) of balanced sets is balanced.*

*iii.* *Let $(A_j)_{j\in\Theta}$ be a chain (no cardinality constraint) of convex sets, that is, for every $j_1$, $j_2\in\Theta$, either $A_{j_1}\subseteq A_{j_2}$ or $A_{j_2}\subseteq A_{j_1}$. Then*

$$\bigcup_{j\in\Theta} A_j$$

*is a convex set.*

*Note that, in general, even the union of two convex sets does not need to be convex.*

**Proof.** ***i.*** Trivial. ***ii.*** We can use **Proposition 1.9**. If $S := \bigcup_{j\in\Theta} S_j$ with each $S_j$ balanced, then given $v\in S$ there exists $j\in\Theta$ such that $v\in S_j$. Since $S_j$ is balanced, we have

$$[-v,v]_{\mathbb{K}}\subseteq S_j.$$

But $S_j\subseteq S$ and we conclude.

***iii.*** Let $\lambda,\mu\geqslant 0$ such that $\lambda+\mu=1$ and let $x,y\in\bigcup_{j\in\Theta} A_j$. Without loss of generality, we can assume that $x\in A_{j_1}$, $y\in A_{j_2}$ and $A_{j_1}\subseteq A_{j_2}$. This implies that $x\in A_{j_2}$ as well. Therefore

$$\lambda x+\mu y\in A_{j_2}\subseteq \bigcup_{j\in\Theta} A_j$$

because $A_{j_2}$ is convex. This concludes the proof. ■

## 1.3 The action of linear maps on balanced sets, absorbing sets and convex sets

It is important to understand how the action of a linear map influence the geometry of the sets so far introduced. **Proposition 1.29** goes in this direction. Before, we need the following simple, but very useful, observation.

**1.27. Lemma.** *Let $X$ and $Y$ be vector spaces over the same field $\mathbb{K}$ and $f: X\to Y$ a linear map. The image of any symmetric segment in $X$ is a symmetric segment in $Y$:*

$$f([-x,x]_{\mathbb{K}}) = [-f(x), f(x)]_{\mathbb{K}} \quad \forall x\in X. \tag{1.16}$$

*In particular, if $y\in f(X)$, then*

$$f([-x,x]_{\mathbb{K}}) = [-y,y]_{\mathbb{K}} \quad \forall x\in f^{-1}(y). \tag{1.17}$$

*Conversely, if $y\in f(X)\subseteq Y$, the inverse image of the symmetric segment $[-y,y]_{\mathbb{K}}$ is the union of the kernel of $f$ and of all segment $[-x,x]_{\mathbb{K}}$ with $x\in f^{-1}(y)$. Precisely, we have*

$$f^{-1}([-y,y]_{\mathbb{K}}) = \left(\bigcup_{x\in f^{-1}(y)} [-x,x]_{\mathbb{K}}\right)\cup(\ker f). \tag{1.18}$$

**1.28. Remark.** Note that, in general, claiming that a function maps lines to lines is weaker than claiming that it maps symmetric segments to symmetric segments.

**Proof.** Let $x \in X$. We have $f([-x, x]_{\mathbb{K}}) = f(\mathbb{D}_\bullet x) = \mathbb{D}_\bullet f(x) = [-f(x), f(x)]_{\mathbb{K}}$.

Conversely, let $y \in f(X)$. From what we just proved, it follows that for any $x \in f^{-1}(y)$ we have

$$f([-x, x]_{\mathbb{K}}) = [-f(x), f(x)]_{\mathbb{K}} = [-y, y]_{\mathbb{K}}$$

so that $[-x, x]_{\mathbb{K}} \subseteq f^{-1}([-y, y]_{\mathbb{K}})$. The arbitariness of $x$ yields $\bigcup_{x \in f^{-1}(y)}[-x, x]_{\mathbb{K}} \subseteq f^{-1}([-y, y]_{\mathbb{K}})$. On the other hand, if $z \in \ker f$, then $f(z) = 0 \in [-y, y]_{\mathbb{K}}$. Overall,

$$(\bigcup_{x \in f^{-1}(y)}[-x, x]_{\mathbb{K}}) \cup (\ker f) \subseteq f^{-1}([-y, y]_{\mathbb{K}}).$$

It remains to prove the opposite inclusion; namely that for every $u \in f^{-1}([-y, y]_{\mathbb{K}})$ either $f(u) = 0$ or there exist $x \in f^{-1}(y)$ and $\sigma \in \mathbb{D}_\bullet$ such that $u = \sigma x$. Let $u \in f^{-1}([-y, y]_{\mathbb{K}})$ and assume $u \notin \ker f$. Then, there exists $0 \neq \sigma \in \mathbb{D}_\bullet$ such that $f(u) = \sigma y$. Hence $f(\sigma^{-1} u) = y$ so that if we set $x := \sigma^{-1} u$ we have $x \in f^{-1}(y)$ and $u = \sigma x$. ■■■■

**1.29. Proposition.** *Let $X$ and $Y$ be vector spaces over the same field $\mathbb{K}$. Let $f$ be a* **linear** *map defined on $X$ and taking values in $Y$. The following assertions hold:*

***i.*** *Let $A \subseteq X$. If $A$ is balanced (resp. convex) in $X$, then $f(A)$ is balanced (resp. convex) in $Y$.*

***ii.*** *If $f$ is* **surjective** *and $A \subseteq X$ is absorbing in $X$, then $f(A)$ is absorbing in $Y$.*

***iii.*** *Let $B \subseteq Y$. If $B$ is balanced (resp. convex, resp. absorbing) in $Y$, then $f^{-1}(B)$ is balanced (resp. convex, resp. absorbing) in $X$.*

**1.30. Remark.** Note that if $f$ is not surjective then $f(A)$ is included in a proper subspace $M$ of $Y$ and, therefore, $f(A)$ cannot be absorbing. In fact, if $f(A)$ is absorbing so has to be any superset of $f(A)$ (cf. Proposition 1.26)

**Proof.** ***i.*** ▶ Since $f$ is linear, due to (1.7), we have $\mathbb{D}_\bullet f(A) = f(\mathbb{D}_\bullet A)$. Also, as $A$ is balanced, we have $\mathbb{D}_\bullet A = A$ and therefore $\mathbb{D}_\bullet f(A) = f(A)$. This shows that $f(A)$ is balanced. ▶ If $A$ is convex then, for every $\lambda, \mu \geqslant 0$ such that $\lambda + \mu = 1$, we have $\lambda f(A) + \mu f(A) = f(\lambda A + \mu A) = f(A)$.

***ii.*** ▶ Let $y \in Y$. Since $f$ is surjective, the set $f^{-1}(y)$ is nonempty. Let $x \in f^{-1}(y)$. By (1.17) we have that $f([-x, x]_{\mathbb{K}}) = [-y, y]_{\mathbb{K}}$. Since $A$ is absorbing, we have $\rho_0[-x, x]_{\mathbb{K}} \subseteq A$ for some sufficiently small $\rho_0 > 0$. By linearity

$$\rho_0[-y, y]_{\mathbb{K}} = \rho_0 f([-x, x]_{\mathbb{K}}) = f(\rho_0[-x, x]_{\mathbb{K}}) \subseteq f(A).$$

***iii.*** ▶ Let $B$ be a balanced set. Then, due to (1.8), $\mathbb{D}_\bullet f^{-1}(B) \subseteq f^{-1}(\mathbb{D}_\bullet B) = f^{-1}(B)$. Hence, $f^{-1}(B)$ is balanced.

▶ Let $B$ be a convex set. Then, for every $\lambda, \mu \geqslant 0$ such that $\lambda + \mu = 1$, we have (cf. (1.8))

$$\begin{aligned} \lambda f^{-1}(B) + \mu f^{-1}(B) &\subseteq f^{-1}(\lambda B) + f^{-1}(\mu B) \\ &\subseteq f^{-1}(\lambda B + \mu B) \\ &= f^{-1}(B). \end{aligned}$$

Hence, $f^{-1}(B)$ is convex.

▶ Pick any arbitrary point $x \in X$. Set $y = f(x)$. Since $B \subseteq Y$ is absorbing, there exists $\rho_0 > 0$ such that $\rho_0[-y, y]_{\mathbb{K}} \subseteq B$. By (1.18) we conclude that

$$\rho_0[-x, x]_{\mathbb{K}} \subseteq f^{-1}(\rho_0[-y, y]_{\mathbb{K}}) \subseteq f^{-1}(B).$$

Hence, $f^{-1}(B)$ is absorbing.

○

## 1.4 Weighted averages in a convex space. The convex hull

**1.31. Definition.** Given a subset $A$ of a vector space $X$, we call **convex hull** (or **convex envelope**) of $A$, the smallest convex subset of $X$ containing $A$. The convex hull always exists because every subset of $X$ is contained in at least a convex set, namely the whole space $X$. Also, since the intersection of any family of convex sets is still a convex set (cf. Proposition 1.26), the minimal convex set (with respect to set inclusion) coincides with the intersection of all convex sets containing $A$. For any $A \subseteq X$ we denote by $\mathrm{Conv}(A)$ or by $K(A)$ the convex hull of $A$. Formally

$$\begin{aligned} K(A) &= \min_{(\subseteq)} \{C \subseteq X :: C \supseteq A \text{ and } C \text{ is convex}\} \\ &= \cap\{C \subseteq X :: C \supseteq A \text{ and } C \text{ is convex}\}. \end{aligned}$$

Note that, if $A$ is convex then $K(A) = A$. Also, note that if $A \subseteq B$ then $K(A) \subseteq K(B)$ because of $\{C_B \supseteq B :: C_B \text{ is convex}\} \subseteq \{C_A \supseteq A :: C_A \text{ is convex}\}$.

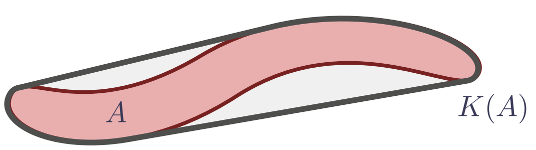



**Figure 1.5.** A generic set $A$ in $\mathbb{R}^2$, and its convex hull $K(A)$.

**1.32. Definition.** Given a generic subset $A$ of a vector space $X$, we say that an element $b \in X$ is a **convex combination** (or a **barycentric combination**, or a **weighted mean**) of elements of $A$, if there exist a finite set of elements $(a_k)_{k\in\mathbb{N}_n} \subseteq A$ and corresponding positive scalars $(\lambda_k)_{k\in\mathbb{N}_n} \geqslant 0$ with $\sum_{k\in\mathbb{N}_n} \lambda_k = 1$ such that $b = \sum_{k\in\mathbb{N}_n} \lambda_k a_k$.

A convex combination can always be understood as a generalized sum $\sum_{a\in A} \lambda(a)\, a$ with non-negative coefficients $(\lambda(a))_{a\in A}$ having **finite support** and such that $\sum_{a\in A} \lambda(a) = 1$, i.e., as a convex combination of all elements of $A$. We denote by $H(A)$ the set consisting of all convex combinations of elements of $A$. In other words:

$$H(A) := \left\{ x \in X :: x = \sum_{a\in A} \lambda(a)\, a, \quad \begin{array}{c} \text{for some } (\lambda(a))_{a\in A}\text{, with finite support} \\ \text{nonnegative, and such that } \sum_{a\in A} \lambda(a) = 1. \end{array} \right\}$$

We refer to $H(A)$ as the **convex span** of $A$. Note that, if $A \subseteq B$, then $H(A) \subseteq H(B)$. In fact, $H(A)$ consists of the elements of $H(B)$ whose coefficients $(\lambda(a))_{a\in A}$ have (finite) support contained in $A$.

**1.33. Proposition.** *Let $X$ be a vector space and $A$ a subset of $X$. The following assertions hold:*

*i. $H(A)$ is a convex set, and it includes $A$. Therefore, by the minimality of $K(A)$, we have $H(A) \supseteq K(A) \supseteq A$.*

*ii. If $A$ is convex then $A \supseteq H(A)$, i.e., $A$ contains all the convex combinations of its points. In particular, owing to i., we get that $H(A) = A$ whenever $A$ is convex.*

*Finally, combining i. and ii., we obtain that*

$$H(A) = K(A)$$

*because of $H(A) \subseteq H(K(A)) = K(A) \subseteq H(A)$.*

**PROOF.** We shall proceed as follows:

1. We prove that if $A$ is convex then $H(A) \subseteq A$. Clearly $A \subseteq H(A)$ and therefore

$$A = H(A) \quad \text{whenever } A \text{ is a convex set.}$$

2. We then prove that $H(A)$ is convex, regardless of whether $A$ is convex or not. By the monotonicity of $H$ and the minimality of $K$ we conclude that $H(A) \subseteq H(K(A)) = K(A) \subseteq H(A)$.

**Step 1.** Assume $A$ **convex**. We have to show that any convex combination $b = \sum_{i \in \mathbb{N}_n} \lambda_i a_i$ of elements of $A$ ($a_i \in A$ and $\lambda_1 + \cdots + \lambda_n = 1$), still belongs to $A$.

We argue by induction on $n$. If $n = 1$ then clearly $b = \lambda_1 a_1 = a_1 \in A$ by assumption. If $n = 2$ then $b = \lambda_1 a_1 + (1 - \lambda_1) a_2 \in A$ because this is nothing but the definition of convex set. Let us consider the case $n \geqslant 3$. We assume that the result is true for any convex combination of $n - 1$ terms and we write

$$b = \sum_{i \in \mathbb{N}_n} \lambda_i a_i = (1 - \lambda_n) \sum_{i \in \mathbb{N}_{n-1}} \frac{\lambda_i}{1 - \lambda_n} a_i + \lambda_n a_n$$

with $1 = \sum_{i \in \mathbb{N}_n} \lambda_i = \lambda_n + \sum_{i \in \mathbb{N}_{n-1}} \lambda_i$. Here, without loss of generality, we are assuming that $\lambda_n \neq 1$ because, if $\lambda_n = 1$, than $b = a_n \in A$ and we are done.

We set $c := \sum_{i \in \mathbb{N}_{n-1}} \frac{\lambda_i}{1 - \lambda_n} a_i$ and note that, by the induction hypothesis, $c \in A$, because it is a convex combination of $n - 1$ elements of $A$ — indeed, $\sum_{i \in \mathbb{N}_{n-1}} \lambda_i = 1 - \lambda_n$. Therefore, $b = (1 - \lambda_n) c + \lambda_n a$ is the convex combination of the two elements $c, a \in A$. Hence, $b \in A$.

**Step 2.** We now show that $H(A)$ is convex. We have to prove that any convex combination of convex combinations of elements of $A$ is still a convex combination of elements of $A$. To this end, we observe that for any $\lambda_1, \lambda_2 \geqslant 0$ such that $\lambda_1 + \lambda_2 = 1$, and for any convex combinations $\sum_{a \in A} \mu_1(a) a$, $\sum_{a \in A} \mu_2(a) a$ we have

$$\lambda_1 \sum_{a \in A} \mu_1(a) a + \lambda_2 \sum_{a \in A} \mu_2(a) a \quad = \quad \sum_{a \in A} (\lambda_1 \mu_1(a) + \lambda_2 \mu_2(a)) a$$

and $\sum_{a \in A} (\lambda_1 \mu_1(a) + \lambda_2 \mu_2(a)) = \lambda_1 + \lambda_2 = 1$. This concludes the proof. ■■■■

**1.34. Corollary.** *The convex hull of a balanced set is balanced as well.* **Formally:** *Let $X$ be a vector space and $A$ a subset of $X$. If $A$ is balanced then $K(A)$ (equivalently $H(A)$) is balanced.*

**Proof.** Let $A$ be a **balanced** set, $K(A)$ its convex hull, and let $x \in K(A)$. Then, $x = \sum_{i \in \mathbb{N}_n} \lambda_i a_i$ for some $a_i \in A$, and $\lambda_i \geqslant 0$ with $\sum_{i \in \mathbb{N}_n} \lambda_i = 1$. Let $\lambda \in \mathbb{K}$ with $|\lambda| \leqslant 1$. One has

$$\lambda x = \sum_{i \in \mathbb{N}_n} \lambda_i \lambda a_i = \sum_{i \in \mathbb{N}_n} \lambda_i b_i \quad \textbf{with} \quad b_i := \lambda a_i.$$

But $b_i \in A$ because $A$ is a balanced set (by assumption). Hence, $\lambda x \in K(A)$.

○

## 1.5 Seminorms on a vector space

**1.35. Definition.** Let $X$ be a vector space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$). A function $\mathfrak{p}: X \to \mathbb{R}_+$ defined on the vector space $X$ is called a **seminorm** if

**SN₁.** $\mathfrak{p}$ is **subadditive**:

$$\mathfrak{p}(x+y) \leqslant \mathfrak{p}(x) + \mathfrak{p}(y) \quad \textbf{for all } x, y \in X; \tag{1.19}$$

**SN₂.** $\mathfrak{p}$ is **circularly homogeneous** (or **absolutely homogeneous**):

$$\mathfrak{p}(\lambda x) = |\lambda| \mathfrak{p}(x) \quad \textbf{for all } \lambda \in \mathbb{K}, x \in X. \tag{1.20}$$

Note that, if $\mathbb{K} = \mathbb{C}$, $\mathfrak{p}$ is not only symmetric with respect to the origin, i.e., $\mathfrak{p}(x) = \mathfrak{p}(-x)$ for any $x \in X$, but even *circularly symmetric*, meaning that $\mathfrak{p}(x) = \mathfrak{p}(\lambda x)$ for every $x \in X$ and any $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$.

The value $\mathfrak{p}(x)$ of $\mathfrak{p}$ at $x \in X$ is often denoted by the symbol $|x|_{\mathfrak{p}}$. In general, cf. **Proposition 1.37**, for a seminorm $\mathfrak{p}$ there holds $\mathfrak{p}(0) = 0$. However, it is not necessarily the case that $\mathfrak{p}(x) = 0$ implies $x = 0$. A seminorm $\mathfrak{p}$, such that $\mathfrak{p}(x) \neq 0$ whenever $x \neq 0$ is called a **norm on** $X$. The set $\ker \mathfrak{p} := \{x \in X :: \mathfrak{p}(x) = 0\}$ is called the kernel of the seminorm $\mathfrak{p}$ and, cf. **Proposition 1.37**, it is a vector subspace of $X$. Clearly, a seminorm is a norm if, and only if, its kernel is the zero vector space.

**Exercise 1.7.** Recall that if $X$ is a vector space over the field $\mathbb{K}$, $f: X \to \mathbb{R}$ a real-valued function, and $k$ an *integer*, then $f$ is said to be **homogeneous of degree $k$** (or $k$-homogeneous) if $f(\lambda x) = \lambda^k f(x)$ for every $\lambda > 0$. Prove that the following result holds.

**1.36. Proposition.** *Let $X$ be a vector space over $\mathbb{K}$, and $f: X \to \mathbb{R}$ a real-valued function. The function $f$ is absolutely homogeneous if, and only if, $f$ is $1$-homogeneous and circular symmetric (i.e., such that $f(\lambda x) = f(x)$ for every $|\lambda| = 1$).*

**Solution.** It is clear that if $f$ is absolutely homogeneous then it is circular symmetric and homogeneous of degree one (because $f(\lambda x) = |\lambda| f(x)$ for any $\lambda \in \mathbb{K}$); however, if $f$ is $1$-homogeneous and $\mathbb{K} = \mathbb{R}$, then, in general, we can only say that

$$f(\lambda x) = \begin{cases} |\lambda| f(x) & \text{if } \lambda \geqslant 0, \\ |\lambda| f(-x) & \text{if } \lambda \leqslant 0. \end{cases}$$

Indeed, for $\lambda \in \mathbb{R}$ we have $f(\lambda x) = f(\operatorname{sign}(\lambda)|\lambda| x) = |\lambda| f(\operatorname{sign}(\lambda) x)$. It follows that, when $\mathbb{K} = \mathbb{R}$, the reverse implication «if $f$ is $1$-homogeneous then $f$ is absolutely homogeneous» holds when $f$ is symmetric with respect to the origin (that is $f(x) = f(-x)$ for every $x \in X$). When $\mathbb{K} = \mathbb{C}$ we need to assume something more, i.e., that $f(\lambda x) = f(x)$ for every $|\lambda| = 1$. Indeed, under this assumption, for any $\lambda \neq 0$ (we can assume that $\lambda \neq 0$ otherwise it is trivially true what we claim) we have

$$f(\lambda x) = |\lambda| f\left(\frac{\lambda}{|\lambda|} x\right) = |\lambda| f(x).$$

This concludes the proof.

**1.37. Proposition.** *Let $\mathfrak{p}$ be a seminorm on a vector space $X$. Then, as a consequence of the absolute homogenéity, $\mathfrak{p}(0) = 0$. Also, the inverse triangular inequality holds*

$$|\mathfrak{p}(x_1) - \mathfrak{p}(x_2)| \leqslant \mathfrak{p}(x_1 - x_2) \quad \textbf{for all } x_1, x_2 \in X. \tag{1.21}$$

*Finally,* $\ker \mathfrak{p}$ *is a linear subspace of* $X$

**1.38. Remark.** The proof never uses the assumption that a seminorm take values in $\mathbb{R}_+$. It follows that if $\mathfrak{p}\colon X \to \mathbb{R}$ is a subadditive and absolutely homogeneous real-valued functional defined on the vector space $X$ then necessarily $\mathfrak{p}(x) \geqslant 0$ for every $x \in X$. This is a consequence of the inverse triangular inequality (1.21): for any $x \in X$ we have $0 \leqslant |\mathfrak{p}(x) - \mathfrak{p}(0)| \leqslant \mathfrak{p}(x - 0) = \mathfrak{p}(x)$. Therefore, in the definition of seminorm, one can relax the condition on the codomain of $\mathfrak{p}$ requiring only that $\mathfrak{p}$ is real-valued.

**Proof.** From the absolute homogenéity of $\mathfrak{p}$ we get $\mathfrak{p}(0_X) = \mathfrak{p}(0_{\mathbb{K}} \cdot 0_X) = |0_{\mathbb{K}}| \cdot \mathfrak{p}(0_X) = 0$. For the inverse triangular inequality, we observe that, due to sublinearity, we have, for any $x_1, x_2 \in X$,

$$\mathfrak{p}(x_1) = \mathfrak{p}(x_2 + (x_1 - x_2)) \leqslant \mathfrak{p}(x_2) + \mathfrak{p}(x_1 - x_2).$$

Hence, $\mathfrak{p}(x_1) - \mathfrak{p}(x_2) \leqslant \mathfrak{p}(x_1 - x_2)$. But then, interchanging the roles of $x_1$ and $x_2$, also $\mathfrak{p}(x_2) - \mathfrak{p}(x_1) \leqslant \mathfrak{p}(x_2 - x_1) = \mathfrak{p}(x_1 - x_2)$. This proves the reverse triangular inequality.

Finally, if $x_1, x_2 \in \ker \mathfrak{p}$ then, for every $\lambda_1, \lambda_2 \in \mathbb{K}$, we have $\mathfrak{p}(\lambda_1 x_1 + \lambda_2 x_2) \leqslant |\lambda_1| \mathfrak{p}(x_1) + |\lambda_2| \mathfrak{p}(x_2) = 0$. The nonnegativity of $\mathfrak{p}$ implies that $\mathfrak{p}(\lambda_1 x_1 + \lambda_2 x_2) = 0$, i.e., $\lambda_1 x_1 + \lambda_2 x_2 \in \ker \mathfrak{p}$.

**Example 1.39.** If $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two seminorms on the vector space $X$, then the join of $\mathfrak{p}_1$ and $\mathfrak{p}_2$, $\mathfrak{p} := \mathfrak{p}_1 \vee \mathfrak{p}_2$, with $\mathfrak{p}_1 \vee \mathfrak{p}_2 := \max\{\mathfrak{p}_1, \mathfrak{p}_2\}$, is still a seminorm on $X$. Moreover, if at least one among $\mathfrak{p}_1$ and $\mathfrak{p}_2$ is a norm, then also $\mathfrak{p}$ is a norm. Indeed, for any $(\lambda, x) \in \mathbb{K} \times X$ we have $\mathfrak{p}(\lambda x) = (|\lambda| \mathfrak{p}_1(x)) \vee (|\lambda| \mathfrak{p}_2(x)) = |\lambda| \, (\mathfrak{p}_1(x) \vee \mathfrak{p}_2(x))$. Therefore, $\mathfrak{p}$ is absolutely homogeneous. Also, let us consider $x$, $y \in X$. We have

$$\mathfrak{p}(x + y) = \mathfrak{p}_1(x + y) \vee \mathfrak{p}_2(x + y),$$

and two possible things can happen, either $\mathfrak{p}(x + y) = \mathfrak{p}_1(x + y)$ or $\mathfrak{p}(x + y) = \mathfrak{p}_2(x + y)$. Let us assume the first circumstance, the other one can be treated in the same way. We have

$$\mathfrak{p}(x + y) = \mathfrak{p}_1(x + y) \leqslant \mathfrak{p}_1(x) + \mathfrak{p}_1(y) \leqslant (\mathfrak{p}_1(x) \vee \mathfrak{p}_2(x)) + (\mathfrak{p}_1(y) \vee \mathfrak{p}_2(y)).$$

Hence, $\mathfrak{p}$ is subadditive. Finally, as $\mathfrak{p}_1(x) \leqslant \mathfrak{p}_1(x) \vee \mathfrak{p}_2(x) = \mathfrak{p}(x)$, if $\mathfrak{p}_1$ is a norm and $\mathfrak{p}(x) = 0$, then $\mathfrak{p}_1(x) = 0$ and therefore $x = 0$. This shows that $\mathfrak{p}$ is a norm if $\mathfrak{p}_1$ is a norm.

**1.40. Remark.** In general, if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two seminorms on the vector space $X$, it is not the case that the meet of $\mathfrak{p}_1$ and $\mathfrak{p}_2$, $\mathfrak{p} := \mathfrak{p}_1 \wedge \mathfrak{p}_2$, with $\mathfrak{p}_1 \wedge \mathfrak{p}_2 := \min\{\mathfrak{p}_1, \mathfrak{p}_2\}$, is a seminorm on $X$. For a counterexample we refer the reader to [Ex. 5, p. 14, in C. Costara and D. Popa, *Exercises in Functional Analysis*, Springer Science & Business Media, **2013**]

**Example 1.41.** Let $C(\Omega, \mathbb{R})$ be the vector space of all continuous functions defined on the nonempty

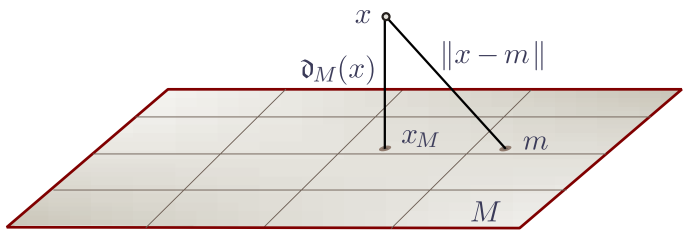



**Figure 1.6.** The functional $\mathfrak{d}_M$ associates to every $x\in X$ its distance from the subspace $M$. It is a seminorm on $X$. In the picture, a schematic representation when $M=\mathbb{R}^2\times\{0\}$, $X=\mathbb{R}^3$ and $\|\cdot\|$ is the euclidean norm. The point $x_M\in M$ denotes the unique point on $M$ such that $\|x-x_M\|=\mathfrak{d}_M(x)$.

open set $\Omega\subseteq\mathbb{R}^N$. Given a compact subset $K\subseteq\Omega$, for any $f\in C(\Omega,\mathbb{R})$ we define the functional

$$\mathfrak{p}_K(f):=\sup_{x\in K}|f(x)|.$$

Clearly, due to Weierstrass extreme value theorem, $\mathfrak{p}_K$ is a well-defined seminorm on $C(\Omega,\mathbb{R})$. But this is not a norm on $C(\Omega,\mathbb{R})$ as $\ker\mathfrak{p}_K\neq\{0\}$. Indeed, $\ker\mathfrak{p}_K$ is the vector subspace of $C(\Omega,\mathbb{R})$ consisting of all functions that are identically zero when restricted to $K$. In particular, if $a\in\Omega$, then $\{a\}$ is a compact subset of $\Omega$ and the kernel of the seminorm $\mathfrak{p}_{\{a\}}\colon f\mapsto|f(a)|$ is the vector subspace of $C(\Omega,\mathbb{R})$ consisting of all functions that vanish at $a\in\Omega$.

**Example 1.42.** Let $L^1_{\mathrm{loc}}(\Omega,\mathbb{R})$ be the vector space of all real-valued locally integrable functions defined on the nonempty open set $\Omega\subseteq\mathbb{R}^N$. For any compact subset $K\subseteq\Omega$ the functional

$$\mathfrak{q}_K(f):=\int_K|f(x)|\,\mathrm{d}x$$

defines a seminorm on $L^1_{\mathrm{loc}}(\Omega,\mathbb{R})$. But it is not a norm as $\ker\mathfrak{q}_K\neq\{0\}$. Indeed $\ker\mathfrak{q}_K$ is the vector subspace of $L^1_{\mathrm{loc}}(\Omega,\mathbb{R})$ consisting of all functions that vanish a.e. on $K$.

**Example 1.43.** Let $(X,\|\cdot\|)$ be a normed space and $M\trianglelefteq X$ a vector subspace of $X$. For any $x\in X$ we define the functional

$$\mathfrak{d}_M(x):=\inf_{m\in M}\|x-m\|=\mathrm{dist}(x,M).$$

The functional $\mathfrak{d}_M$ associates to every $x\in X$ its distance from the subspace $M$. It is a seminorm on $X$. Indeed: ▶ [**absolute homogenêity**] For $\lambda=0$ the homogeneity relation is trivial because $\mathfrak{d}_M(0)=\inf_{m\in M}\|m\|=0$. On the other hand, for any $\lambda\in\mathbb{K}$, $\lambda\neq0$, the map

$$p\in M\mapsto m:=\lambda p\in M$$

gives a parametrization of $M$, so that

$$\mathfrak{d}_M(\lambda x)=\inf_{m\in M}\|\lambda x-m\|=\inf_{p\in M}\|\lambda x-\lambda p\|=|\lambda|\mathfrak{d}_M(x).$$

▶ [**subadditivity**] for any $x_1,x_2\in X$, observing that $(m_1,m_2)\in M\times M\mapsto m_1+m_2\in M$ is a parameterization of $M$, and using the subadditivity of the norm $\|\cdot\|$, we get:

$$\begin{aligned}\mathfrak{d}_M(x_1+x_2) &= \inf_{m\in M}\|x_1+x_2-m\|\\ &= \inf_{(m_1,m_2)\in M\times M}\|x_1+x_2-m_1-m_2\|\end{aligned}$$

$$\begin{aligned}&\leqslant \inf_{m_1\in M}\|x_1-m_1\|+\inf_{m_2\in M}\|x_2-m_2\|\\ &= \mathfrak{d}_M(x_1)+\mathfrak{d}_M(x_2).\end{aligned}$$

However, apart from the case where $M$ is the trivial subspace $M=\{0\}$, the functional $\mathfrak{d}_M$ is never a norm on $X$ because $\ker\mathfrak{d}_M\neq\{0\}$. Indeed $\ker\mathfrak{d}_M=\{x\in X :: \operatorname{dist}(x,M)=0\}$ and since $\operatorname{dist}(x,M)=0$ if, and only if $x\in\bar{M}$, we conclude that $\ker\mathfrak{d}_M=\bar{M}$.

From the previous considerations, the following result easily follows. The functional $x\mapsto\operatorname{dist}(x,M)$ induces a natural seminorm $[\mathfrak{d}_M]$ on the quotient vector space $X/M$:

$$[x]_M\in X/M\mapsto[\mathfrak{d}_M]([x]_M):=\operatorname{dist}(x,M).$$

The map is well-defined because if $x_1,x_2\in[x]_M$ then $x_1-x_2\in M$ and therefore

$$\operatorname{dist}(x_1,M)=\inf_{m\in M}\|x_2+((x_1-x_2)-m)\|=\inf_{p\in M}\|x_2+p\|=\inf_{p\in M}\|x_2-p\|=\operatorname{dist}(x_2,M)$$

the equality $\inf_{p\in M}\|x_2+p\|=\inf_{p\in M}\|x_2-p\|$ holding because $M$ is a symmetric set (in fact, a vector space).

○

⋯

**Example 1.44.** Consider a normed vector space $(X,\|\cdot\|)$. The class $\mathfrak{X}$ of all Cauchy sequence $x_\bullet\colon\mathbb{N}\to X$ in $X$ can be structured in a natural way into a vector space. We can define on $\mathfrak{X}$ a seminorm $\mathfrak{p}_{\mathfrak{X}}$ through the relation

$$\mathfrak{p}_{\mathfrak{X}}(x_\bullet)=\lim_{j\to\infty}\|x_j\|,$$

with $x_\bullet:=(x_j)_{j\in\mathbb{N}}$. This is a well-defined seminorm because from the inverse triangular inequality we have that

$$|\|x_m\|-\|x_n\||\leqslant\|x_m-x_n\|,$$

so that $\|x_j\|$ is a Cauchy sequence in $\mathbb{R}$. It easily checked that this defined aseminorm. The kernel of $\mathfrak{p}_{\mathfrak{X}}$ is the vector subspace of $\mathfrak{X}$ consisting of all sequences in $X$ which converge to zero. ⋯

**1.45. Definition.** Given a seminorm $\mathfrak{p}$ on a vector space $X$, the sets

$$B_\circ:=\{x\in X :: \mathfrak{p}(x)<1\}\quad\textbf{and}\quad B_\bullet:=\{x\in X :: \mathfrak{p}(x)\leqslant 1\}$$

are called, respectively, the **open** (or **unachieved**) **unit semiball** of $\mathfrak{p}$ and the **closed** (or **achieved**) **unit semiball** of $\mathfrak{p}$. Sometimes, when dealing with more than one seminorm, we write $B_\circ(\mathfrak{p})$ and $B_\bullet(\mathfrak{p})$ to avoid ambiguities.

**1.46. Remark.** The qualifications *open* and *closed* in this context are not topological. In fact, we are in a purely algebraic setting. Besides, even when $X$ is a topological space, it is not necessarily the case that $\overline{B_\circ}=B_\bullet$, i.e., that the topological closure of the open unit semiball **of** $\mathfrak{p}$ coincides with the closed unit semiball **of** $\mathfrak{p}$.

○

### 1.5.1. Properties of the semiballs

Let us prove the following result.

**1.47. Proposition.** *Let $X$ be a vector space over the field $\mathbb{K}$ ($\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$) and $\mathfrak{p}$ a seminorm on $X$. The open unit semiball $B_\circ$ of $\mathfrak{p}$ and the closed unit semiball $B_\bullet$ of $\mathfrak{p}$ have the following properties:*

*i.* *They are balanced.*

*ii.* *For any $\lambda \in \mathbb{K}$, $\lambda \neq 0$,*

$$\lambda B_\circ \equiv \{y \in X :: \mathfrak{p}(y) < |\lambda|\} \quad \textbf{and} \quad \lambda B_\bullet \equiv \{y \in X :: \mathfrak{p}(y) \leqslant |\lambda|\}.$$

*iii.* *They are absorbing.*

*iv.* *They are convex.*

*We say that $\lambda B_\circ$ (resp. $\lambda B_\bullet$) is the open semiball (resp. the closed semiball) of $\mathfrak{p}$ of* **radius** $|\lambda|$.

**1.48. Remark.** The assertion *ii.* does not hold when $\lambda=0$. Indeed, when $\lambda=0$ we have $\{0\} = \lambda B_\circ \neq \{y \in X :: \mathfrak{p}(y) < |\lambda|\} = \emptyset$. Also, for what concerns the closed semiball, in general we have $\{0\} = \lambda B_\bullet \subseteq \{y \in X :: \mathfrak{p}(y) \leqslant |\lambda|\} = \ker \mathfrak{p}$.

**PROOF.** *i.* To prove that $B_\circ$ is balanced (the same argument adapts to $B_\bullet$) we need to prove (by definition) that if $x \in B_\circ$ then $[-x,x]_{\mathbb{K}} \subseteq B_\circ$. But this is trivial because, for any $|\lambda| \leqslant 1$ we have $\mathfrak{p}(\lambda x) = |\lambda|\mathfrak{p}(x) \leqslant \mathfrak{p}(x) < 1$.

*ii.* By **Proposition 1.24**, having already proved that $B_\circ$ and $B_\bullet$ are balanced, we know that $\lambda B_\circ = |\lambda| B_\circ$ and $\lambda B_\bullet = |\lambda| B_\bullet$ for every $\lambda \in \mathbb{K}$. It is therefore sufficient to prove that $|\lambda| B_\circ \equiv \{x \in X :: \mathfrak{p}(x) < |\lambda|\}$.

Let $y = |\lambda| x \in |\lambda| B_\circ$. We then have $\mathfrak{p}(x) < 1$ and $\mathfrak{p}(y) = |\lambda|\mathfrak{p}(x) < |\lambda|$. On the other hand, if $\mathfrak{p}(y) < |\lambda|$ then $\mathfrak{p}(y/|\lambda|) < 1$; thus $y/|\lambda| \in B_\circ$ and, therefore, $y = |\lambda|(y/|\lambda|) \in |\lambda| B_\circ$. Similar argument holds for $B_\bullet$.

*iii.* To prove that $B_\circ$ and $B_\bullet$ are absorbing it is sufficient to focus on $B_\circ$ because $B_\circ \subseteq B_\bullet$ and every superset of an absorbing set is still absorbing.

Note that every superset of an absorbing set is still absorbing

Having alredy proved that $B_\circ$ is balanced, according to **Proposition 1.24**, to prove that $B_\circ$ is absorbing it is sufficent to show that for any $x \in X$ there exists $\rho_0(x) > 0$ such that $\rho_0(x)x \in B_\circ$, i.e., such that $\mathfrak{p}(\rho_0(x)x) < 1$. This amounts to the fulfillment of the condition

$$\rho_0(x)\mathfrak{p}(x) < 1.$$

It is clear that if $\mathfrak{p}(x) = 0$ or $|\lambda| = 0$ every $\rho_0(x)$ works, otherwise it is sufficient to take

$$0 < \rho_0(x) < \frac{1}{\mathfrak{p}(x)}.$$

○

*iv.* Eventually, the inequality $\mathfrak{p}(\lambda x + \mu y) \leqslant |\lambda|\mathfrak{p}(x) + |\mu|\mathfrak{p}(y)$ shows that both $B_\circ$ and $B_\bullet$ are convex. Indeed, with reference to $B_\circ$, if $\mathfrak{p}(x)$ and $\mathfrak{p}(y)$ are both less than one and $|\lambda| + |\mu| = 1$, then $\mathfrak{p}(|\lambda| x + |\mu| y) < |\lambda| + |\mu| = 1$. The same argument holds for $B_\bullet$. ■

The following observation will be useful when we will talk about locally convex spaces. It will be recalled in **Remark 4.13**.

**1.49. Proposition.** *Let $\mathfrak{p}_\alpha, \mathfrak{p}_\beta$ be two seminorms on the same vector space $X$. If $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_\beta$ then $B_\bullet(\mathfrak{p}_\alpha) \subseteq B_\bullet(\mathfrak{p}_\beta)$ and vice versa. In symbols:*

$$\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_\beta \quad \Leftrightarrow \quad B_\bullet(\mathfrak{p}_\alpha) \subseteq B_\bullet(\mathfrak{p}_\beta)$$

*In particular, if we denote by $\lambda\mathfrak{p}$, $\lambda \geqslant 0$, the seminorm $x \mapsto \lambda\mathfrak{p}(x)$ then, for any $\lambda, \mu \geqslant 0$, there holds $B_\bullet(\lambda\mathfrak{p}) \subseteq B_\bullet(\mu\mathfrak{p})$ if, and only if, $\lambda \geqslant \mu \geqslant 0$.*

*Eventually, we note that for any $\lambda \in \mathbb{K} \setminus \{0\}$ one has*

$$B_\bullet(|\lambda|\mathfrak{p}) = \lambda^{-1} B_\bullet(\mathfrak{p}) = |\lambda|^{-1} B_\bullet(\mathfrak{p})$$

**PROOF.** The $\Rightarrow$ implication is trivial. For the other direction, consider $x \in B_\bullet(\mathfrak{p}_\alpha)$. Given any $\varepsilon > 0$, one has $\mathfrak{p}_\alpha(x/(\varepsilon + \mathfrak{p}_\alpha(x))) < 1$. Hence $x/(\varepsilon + \mathfrak{p}_\alpha(x)) \in B_\circ(\mathfrak{p}_\alpha)$. By hypothesis, this implies $x/(\varepsilon + \mathfrak{p}_\alpha(x)) \in B_\circ(\mathfrak{p}_\beta)$, that is, $\mathfrak{p}_\beta(x/(\varepsilon + \mathfrak{p}_\alpha(x))) < 1$. It follows that $\mathfrak{p}_\beta(x) < \varepsilon + \mathfrak{p}_\alpha(x)$. By the arbitrariness of $\varepsilon$ we get that $\mathfrak{p}_\beta(x) \leqslant \mathfrak{p}_\alpha(x)$. To conclude, note that if $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_\beta$ then, necessarily, $\ker \mathfrak{p}_\alpha \trianglelefteq \ker \mathfrak{p}_\beta$.

We have to consider the quantity $\varepsilon + \mathfrak{p}_\alpha(x)$ because the kernels of $\mathfrak{p}_\alpha$ and $\mathfrak{p}_\beta$ can be nontrivial.

Next, note that

$$\begin{aligned} B_\bullet(|\lambda|\mathfrak{p}) &= \{x \in X :: |\lambda|\mathfrak{p}(x) \leqslant 1\} \\ &= \{\lambda^{-1}x \in X :: |\lambda|\mathfrak{p}(\lambda^{-1}x) \leqslant 1\} \\ &= \{\lambda^{-1}x \in X :: \mathfrak{p}(x) \leqslant 1\} \\ &= \lambda^{-1} B_\bullet(\mathfrak{p}). \end{aligned}$$

Eventually, since $B_\bullet(\mathfrak{p})$ is balanced (cf. **Proposition 1.47**), from **Proposition 1.24** we get $\lambda^{-1}B_\bullet(\mathfrak{p}) = |\lambda|^{-1}B_\bullet(\mathfrak{p})$. This concludes the proof. ■■■■

### 1.5.2. Gauge of a set (Minkowski functional)

Seminorms on vector spaces belong to a very important class of functionals, called **Minkowski functionals**, that allow building seminorms via suitable subsets of the ambient vector space.

Hermann Minkowski (22 June 1864 – 12 January 1909) was a German mathematician and professor at Königsberg, Zürich and Göttingen.

**1.50. Definition.** Let $X$ be a vector space and $A \subseteq X$ a **generic** subset. The map $\mathfrak{p}_A\colon X \to [0, +\infty]$ defined, for any $x \in X$, by

$$\mathfrak{p}_A(x) := \inf\{\alpha \in \mathbb{R}_+^* :: x \in \alpha A\} \quad \textbf{with} \quad \mathbb{R}_+^* = ]0, +\infty[,$$

is called the **gauge of** $A$ (or **the Minkowski functional induced by** $A$). Here, we assume the usual convention $\inf \emptyset = +\infty$. Note that $\mathfrak{p}_A(x) \leqslant 1$ for every $x \in A$.

Note that this is more than a convention. Indeed, any real number is a lower bound for the empty set. Hence, the g.l.b is $+\infty$.

For $A \subseteq X$ and $x \in X$, it is convenient to denote by $[x]_A$ the numerical set

$$[x]_A := \{\alpha \in \mathbb{R}_+^* :: x \in \alpha A\}.$$

Then, $\mathfrak{p}_A(x) = \inf [x]_A$ for every $x \in X$. Also, since $[x]_A \subseteq [x]_B$ when $A \subseteq B$, it follows that

$$\mathfrak{p}_B \preccurlyeq \mathfrak{p}_A \quad \text{when} \quad A \subseteq B.$$

Although we defined the Minkowski functional on a generic subset of $X$, it becomes interesting when $A$ is a convex set, and/or a balanced set, and/or an absorbing set.

The intuition here goes like this. If $\alpha A$ is the open/closed ball of radius $\alpha$, then $\mathfrak{p}_A(x)$ is the "smallest" radius such that $x \in \alpha A$. In other terms, one shrinks the ball by a factor $\alpha$ until the boundary of the ball passes through $x$. The radius $\alpha(x)$ such that the boundary of $A$ passes through $x$ is $\mathfrak{p}_A(x)$.

○

The link between Minkowski functionals and seminorms is the object of the next result.

**1.51. Proposition.** *Let $X$ be a vector space. The following assertions hold:*

*i.* *If $B_\circ(\mathfrak{p})$ (resp. $B_\bullet(\mathfrak{p})$) is the* **open (or closed) unit semiball** *$B_\circ(\mathfrak{p})$ (or $B_\bullet(\mathfrak{p})$) of a seminorm $\mathfrak{p}$ on $X$, then the gauge of $B_\circ(\mathfrak{p})$ (resp. $B_\bullet(\mathfrak{p})$) coincides with $\mathfrak{p}$. In other terms:*

$$\mathfrak{p} \equiv \mathfrak{p}_{B_\circ(\mathfrak{p})} \equiv \mathfrak{p}_{B_\bullet(\mathfrak{p})}$$

*ii.* *If $A$ is an absorbing, balanced, and convex subset of $X$, then the gauge $\mathfrak{p}_A$ induced by $A$ is a* **seminorm**. *But, in general, the closed unit semiball of $\mathfrak{p}_A$ contains $A$, while the open unit semiball of $\mathfrak{p}_A$ is contained in $A$. In other words, if $B_\circ(\mathfrak{p}_A)$ and $B_\bullet(\mathfrak{p}_A)$ are, respectively, the open and closed unit semiballs of (the seminorm) $\mathfrak{p}_A$ then*

$$B_\circ(\mathfrak{p}_A) \subseteq A \subseteq B_\bullet(\mathfrak{p}_A).$$

*Moreover, if $B$ is any set in between $B_\circ(\mathfrak{p}_A)$ and $B_\bullet(\mathfrak{p}_A)$, i.e., if $B_\circ(\mathfrak{p}_A) \subseteq B \subseteq B_\bullet(\mathfrak{p}_A)$ then*

$$\mathfrak{p}_A \equiv \mathfrak{p}_B, \quad \text{in particular } \mathfrak{p}_A \equiv \mathfrak{p}_{B_\bullet(\mathfrak{p}_A)} \equiv \mathfrak{p}_{B_\circ(\mathfrak{p}_A)}.$$

*iii.* *Also, since the* **open (or closed) unit semiball** *$B_\circ(\mathfrak{p})$ (or $B_\bullet(\mathfrak{p})$) of a seminorm $\mathfrak{p}$ on $X$ is an absorbing, balanced, and convex subset of $X$, we have, by* ***i.***, *that $\mathfrak{p} \equiv \mathfrak{p}_{B_\circ(\mathfrak{p})} \equiv \mathfrak{p}_{B_\bullet(\mathfrak{p})}$ and if $B \subseteq X$ is such that $B_\circ(\mathfrak{p}) \subseteq B \subseteq B_\bullet(\mathfrak{p})$*

$$\mathfrak{p}_B \equiv \mathfrak{p} \equiv \mathfrak{p}_{B_\circ(\mathfrak{p})} \equiv \mathfrak{p}_{B_\bullet(\mathfrak{p})}.$$

**1.52. Remark.** Roughly speaking, the idea to keep in mind is that if $A$ is an absorbing, balanced, and convex subset of $X$, then the gauge $\mathfrak{p}_A$ of $A$ is a seminorm. But in general, this seminormdoes not retain full information about $A$, in the sense that if we only know $\mathfrak{p}_A$ then we do not know if $\mathfrak{p}_A$ come from $A$ or any other subset $B$ in between $B_\circ(\mathfrak{p}_A)$ and $B_\bullet(\mathfrak{p}_A)$. Later on (cf. **Proposition 3.65**), when we will introduce topological vector spaces, we will see that if $A$ is also closed, then $A$ can be recovered through its gauge, because, in that case, $\mathfrak{p}_A = B_\bullet(\mathfrak{p}_A)$.

**PROOF.** ***i.*** If $A = B_\circ(\mathfrak{p})$ is the open unit semiball of a seminorm $\mathfrak{p}$, then (cf. **Proposition 1.47**, ***ii.***)

Recall that $\lambda B_\circ \equiv \{\mathfrak{p}(y) < |\lambda|\}$ and $\lambda B_\bullet \equiv \{\mathfrak{p}(y) \leqslant |\lambda|\}$ for any $\lambda \in \mathbb{K}$.

$$\mathfrak{p}_A(x) = \inf\{\alpha \in \mathbb{R}_+^* :: x \in \alpha B_\circ(\mathfrak{p})\} = \inf\{\alpha \in \mathbb{R}_+^* :: \alpha > \mathfrak{p}(x)\} = \mathfrak{p}(x).$$

Similarly, if $A = B_\bullet(\mathfrak{p})$ is the closed unit semiball of a seminorm $\mathfrak{p}$, then (cf. **Proposition 1.47**, ***ii.***)

Recall that $\lambda B_\circ \equiv \{\mathfrak{p}(y) < |\lambda|\}$ and $\lambda B_\bullet \equiv \{\mathfrak{p}(y) \leqslant |\lambda|\}$ for any $\lambda \in \mathbb{K}$.

$$\mathfrak{p}_A(x) = \inf\{\alpha \in \mathbb{R}_+^* :: x \in \alpha B_\bullet(\mathfrak{p})\} = \inf\{\alpha \in \mathbb{R}_+^* :: \alpha \geqslant \mathfrak{p}(x)\} = \mathfrak{p}(x).$$

***ii.*** Let us show that the gauge of a convex, balanced, and absorbing set is a seminorm.

Since $A$ is absorbing, $\mathfrak{p}_A$ is **finite** for every $x \in X$. In other terms, $\mathfrak{p}_A$ takes values in $\mathbb{R}_+$ (no more in $\bar{\mathbb{R}}_+$).

The fact that the gauge $\mathfrak{p}_A$ is **absolutely homogeneous** is a consequence of $A$ being a balanced set. Indeed, for a generic subset $A$ one always has $\mathfrak{p}_A(\lambda x) = \lambda \mathfrak{p}_A(x)$ for any $\lambda > 0$, because the conditions $\lambda x \in \alpha A$ and $x \in \lambda^{-1}\alpha A$ are equivalent. In other words, $\mathfrak{p}_A$ defines a $1$-homogenous function because:

$$\mathfrak{p}_A(\lambda x) = \inf\{\alpha \in \mathbb{R}_+^* :: x \in \lambda^{-1}\alpha A\} = \lambda \inf\{\lambda^{-1}\alpha \in \mathbb{R}_+^* :: x \in \lambda^{-1}\alpha A\} = \lambda \mathfrak{p}_A(x).$$

To show that $\mathfrak{p}_A$ is absolutely homogeneous, it remains to prove that $\mathfrak{p}_A(\lambda x) = \mathfrak{p}_A(x)$ for any $\lambda \in \mathbb{K}$ such that $|\lambda| = 1$. In fact, after that, for any $\lambda \in \mathbb{K}$ different from $0_{\mathbb{K}}$ (the case $\lambda = 0_{\mathbb{K}}$ is trivial) we have $\mathfrak{p}_A(\lambda x) = |\lambda|\, \mathfrak{p}_A\left(\frac{\lambda}{|\lambda|}x\right) = |\lambda|\, \mathfrak{p}_A(x)$. To prove that $\mathfrak{p}_A(\lambda x) = \mathfrak{p}_A(x)$ when $|\lambda| = 1$, we observe that $\mathfrak{p}_A(\lambda x) = \inf\{\alpha \in \mathbb{R}_+^* :: x \in (\alpha/\lambda)A = \alpha A\} = \mathfrak{p}_A(x)$ because $A$ is balanced (cf. **Proposition 1.24.*i***).

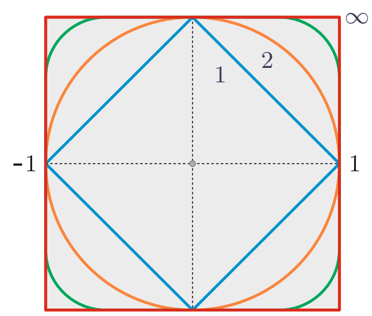



**Figure 1.7.** The open (or closed) unit semiball of a normed vector space uniquely characterizes the norm. Here, it is depicted the shape of the unit balls associated with the $p$-norms in $\mathbb{R}^2$.

Let us show the **subadditivity**. To prove this we need the convexity of $A$. Let $x, y \in X$. First, we prove the following **Claim:** *for any $\beta > \mathfrak{p}_A(x)$ and any $\gamma > \mathfrak{p}_A(y)$ we have*

$$\mathfrak{p}_A(x+y) < \beta + \gamma.$$

After that, it will be sufficient to define, for any $\varepsilon > 0$, the families $\beta_\varepsilon := \mathfrak{p}_A(x) + \varepsilon$ and $\gamma_\varepsilon := \mathfrak{p}_A(y) + \varepsilon$ to infer that

$$\mathfrak{p}_A(x+y) < \mathfrak{p}_A(x) + \mathfrak{p}_A(y) + 2\varepsilon \quad \textbf{for every } \varepsilon > 0.$$

Letting $\varepsilon \to 0$ we obtain the subadditivity of $\mathfrak{p}_A$.

To prove the claim, we can focus on the case in which both $\mathfrak{p}_A(x)$ and $\mathfrak{p}_A(y)$ are finite. We note that if $\beta > \mathfrak{p}_A(x)$ and $\gamma > \mathfrak{p}_A(y)$, then, from the definition of the gauge functional $\mathfrak{p}_A$, there exist $\beta > \beta_* > 0$ and $\gamma > \gamma_* > 0$ such that $x \in \beta_* A$ and $y \in \gamma_* A$. But then, $x + y \in (\beta_* A + \gamma_* A) = (\beta_* + \gamma_*)A$ because $A$ is a convex set. This means that $\beta_* + \gamma_* \in \{\alpha \in \mathbb{R}_+^* :: x + y \in \alpha A\}$. Therefore, $\mathfrak{p}_A(x+y) = \inf\{\alpha \in \mathbb{R}_+^* :: x + y \in \alpha A\} \leqslant \beta_* + \gamma_* < \beta + \gamma$.

Next, we show that $B_\circ(\mathfrak{p}_A) \subseteq A \subseteq B_\bullet(\mathfrak{p}_A)$. This is readily seen. Indeed, on the one hand, $x \in A \Rightarrow \mathfrak{p}_A(x) \leqslant 1 \Rightarrow x \in B_\bullet(\mathfrak{p}_A)$. On the other hand, $x \in B_\circ(\mathfrak{p}_A) \Rightarrow \mathfrak{p}_A(x) < 1$ which means that $x \in \alpha A$ for some $0 < \alpha < 1$. Since $A$ is balanced $x \in \lambda A$ for every $\lambda > \alpha$, in particular for $\lambda = 1$.

Moreover, if $B$ is any set in between $B_\circ(\mathfrak{p}_A)$ and $B_\bullet(\mathfrak{p}_A)$, i.e., if $B_\circ(\mathfrak{p}_A) \subseteq B \subseteq B_\bullet(\mathfrak{p}_A)$ then

$$\mathfrak{p}_{B_\bullet(\mathfrak{p}_A)} \preccurlyeq \mathfrak{p}_B \preccurlyeq \mathfrak{p}_{B_\circ(\mathfrak{p}_A)}.$$

But $\mathfrak{p}_A$ is a seminorm and, therefore, by ***i.***, $\mathfrak{p}_A \equiv \mathfrak{p}_{B_\circ(\mathfrak{p}_A)} \equiv \mathfrak{p}_{B_\bullet(\mathfrak{p}_A)}$. Hence, $\mathfrak{p}_A = \mathfrak{p}_B$.

***iii.*** Also, since the **open (or closed) unit semiball** $B_\circ(\mathfrak{p})$ (or $B_\bullet(\mathfrak{p})$) of a seminorm $\mathfrak{p}$ on $X$ is an absorbing, balanced, and convex subset of $X$, we have, by ***i.***, that $\mathfrak{p} \equiv \mathfrak{p}_{B_\circ(\mathfrak{p})} \equiv \mathfrak{p}_{B_\bullet(\mathfrak{p})}$. But now, if $B \subseteq X$ is such that $B_\circ(\mathfrak{p}) \subseteq B \subseteq B_\bullet(\mathfrak{p})$, then $\mathfrak{p}_{B_\bullet(\mathfrak{p})} \preccurlyeq \mathfrak{p}_B \preccurlyeq \mathfrak{p}_{B_\circ(\mathfrak{p})}$ and, therefore

$$\mathfrak{p}_B \equiv \mathfrak{p} \equiv \mathfrak{p}_{B_\circ(\mathfrak{p})} \equiv \mathfrak{p}_{B_\bullet(\mathfrak{p})}.$$

**1.53. Corollary.** *Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two seminorms defined on the same vector space $X$ and having the* **same closed unit semiball** (*or the* **same open unit semiball**). *Then, the two seminorms are identical:* $\mathfrak{p}_1 \equiv \mathfrak{p}_2$.

**Proof.** The assertion follows by transitivity. Indeed, according to the previous **Proposition 1.51**, both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ coincide with the Minkowski gauge $\mathfrak{p}_A$.

**1.54. Remark.** **Corollary 1.53** tells us, in particular, that the open (or closed) unit semiball of a normed vector space uniquely characterizes the norm. That is the reason why is so "famous" the usual picture reported in **Figure 1.7** that depicts the shape of the unit balls associated with the $p$-norms in euclidean spaces.

## 1.6 Generalized sequences (nets)

A (**partial**) **order** (relation) on a set $A$, usually denoted by the symbols $\leq$ or $\preccurlyeq$, is a **binary relation** on $A$ having the following properties: **1)** $a \preccurlyeq a$ (**reflexivity**); **2)** if $a \preccurlyeq b$ and $b \preccurlyeq c$, then $a \preccurlyeq c$ (**transitivity**); **3)** if $a \preccurlyeq b$ and $b \preccurlyeq a$, then $a = b$ (**anti-symmetry**). If $\preccurlyeq$ is an order, then the relation $\prec$ defined by $a \prec b$ when $a \preccurlyeq b$ and $a \neq b$ is called **a strict order**. A strict order can be defined as a transitive relation such that $a \prec b$ and $b \prec a$ cannot occur simultaneously, i.e., if $a \prec b$ occurs then $b \not\prec a$. The expression $a \preccurlyeq b$ is usually read as "*$a$ is less than or equal to $b$*" or "*$b$ is greater than or equal to $a$*", and $a \prec b$ is read as "*$a$ is less than $b$*" or "*$b$ is greater than $a$*". The order is called **total** if for any $a$, $b \in A$ either $a \preccurlyeq b$ or $b \preccurlyeq a$, i.e., when any pair of elements from $A$ is **comparable**.

A total order is also called a *linear* order, and a set equipped with a total order is sometimes called a **chain** or a **totally ordered set**. For emphasis, an order which is not (necessarily) total is referred to as a **partial order**.

Let $E$ be a partially ordered set and $A$ a subset of $E$. We say that $a_* \in A$ is a **minimum** of $A$ (resp. a **maximum**) of $A$ if $a_* \preccurlyeq a$ (resp. if $a \preccurlyeq a_*$) for every $a \in A$. In general, given a subset $A \subseteq E$, it is not the case that $A$ admits a minimum (resp. a maximum). But if it exists then it is unique and denoted by $\min_{(\preccurlyeq)} A$ (resp. $\max_{(\preccurlyeq)} A$).

An element $m \in E$ is called an **upper bound** for $A$ if $a \preccurlyeq m$ for every $a \in A$. In this case, to shorten notation, we sometimes write

$$A \preccurlyeq m. \tag{1.22}$$

We say that $m$ is the **least upper bound** of $A$ in $E$, and we denote it by $\sup A$ (if it exists), when $m$ is the minimum among all the upper bounds of $A$. That is, when $m \in E$ is such that the following property holds:

$$A \preccurlyeq m \textbf{ and } \forall y \in E (A \preccurlyeq y \Rightarrow A \preccurlyeq m \preccurlyeq y). \tag{1.23}$$

If $\sup A$ exists it is unique.

A partially ordered set $(E, \preccurlyeq)$ is said to be a **join-semilattice** (or **reticulated to the right**) if every couple $(a, b)$ of elements of $A$ admits a **join**: $a \vee b := \sup \{a, b\}$. We set $a \wedge b := \inf \{a, b\}$. The operator $\wedge$ is called the **meet**. The dual notions of **lower bound**, **greatest lower bound**, **meet-semilattice** (or **reticulated to the left**) are defined in an obvious similar way.

### 1.6.1. Directed Sets

**1.55. Definition.** Let $(\Lambda, \preccurlyeq)$ be a partially ordered set. We say that $\Lambda$ is **(upward) directed by** $\preccurlyeq$, or **filtered to the right** by $\preccurlyeq$, if every pair of elements of $\Lambda$ admits an upper bound.

Note that if $(\Lambda, \preccurlyeq)$ is a directed set, then any *finite* subset of $\Lambda$ has an upper bound. Indeed, if $\{\lambda_1, \ldots, \lambda_n\} \subseteq \Lambda$, then

$$\begin{aligned}
\exists \mu_2 \in \Lambda &:: \lambda_1 \preccurlyeq \mu_2 \text{ and } \lambda_2 \preccurlyeq \mu_2,\\
\exists \mu_3 \in \Lambda &:: \mu_2 \preccurlyeq \mu_3 \text{ and } \lambda_3 \preccurlyeq \mu_3,\\
&\vdots\\
\exists \mu_n \in \Lambda &:: \mu_{n-1} \preccurlyeq \mu_n \text{ and } \lambda_n \preccurlyeq \mu_n.
\end{aligned}$$

Thus, in $n-1$ steps we get the existence of $\mu_n \in \Lambda$ such that $\lambda_i \leqslant \mu_n$ for every $i \in \mathbb{N}_n$.

**Example 1.56.** Every totally ordered set is directed, e.g., $(\mathbb{N}, \leqslant)$, $(\mathbb{N}^*, \leqslant)$, $(\mathbb{Q}, \leqslant)$, $(\mathbb{R}, \leqslant)$. More generally, every join-semilattice $(E, \preccurlyeq)$ is directed because for any $(a, b) \in E \times E$, the join $a \vee b$ is an upper bound for both $a$ and $b$. ⋯

**Example 1.57.** Let $(\Lambda, \preccurlyeq)$ be a partially ordered set. If $\max \Lambda$ exists, then $\Lambda$ is a directed set. ⋯

**Example 1.58.** Given a set $\Omega$, the power set $\wp(\Omega)$ of $\Omega$ can be directed in two natural ways. It can be partially ordered by the inclusion relation $\subseteq$, but also via the inverse inclusion relation $\supseteq$. For future uses it is a good idea to stress some aspects of this example.

The partially ordered set $(\wp(\Omega), \subseteq)$ is a **join**-semilattice because $A \vee B = A \cup B$ for every $A$, $B \in \wp(\Omega)$. Indeed,

$$A \cup B \equiv \min_{\subseteq} \{U \in \wp(\Omega) :: A \subseteq U, B \subseteq U\}.$$

It is convenient to say that $A \cup B$ is *the set that looses against any set that wins against* $A$ *and* $B$, *where here to win means to include*. Note that here $\Omega$ wins against any set.

Similarly $(\wp(\Omega), \supseteq)$ is a join-semilattice because now $A \vee B = A \cap B$ for every $A, B \in \wp(\Omega)$. Indeed,

$$\begin{aligned} A \cap B &\equiv \min_{\supseteq} \{U \in \wp(\Omega) :: A \supseteq U, B \supseteq U\} \\ &= \max_{\subseteq} \{U \in \wp(\Omega) :: A \supseteq U, B \supseteq U\}. \end{aligned}$$

It is convenient to say that $A \cap B$ is *the set that looses against any set that wins against* $A$ *and* $B$, *where here to win means to be included.* Note that here $\emptyset$ wins against any set. ⋯

**1.59. Proposition.** *If* $(\Lambda, \subseteq)$ *and* $(\Theta, \sqsubseteq)$ *are two directed sets, the product set* $\Lambda \times \Theta$ *can be directed by the order relation*

$$(\lambda_1, \theta_1) \preccurlyeq (\lambda_2, \theta_2) \quad \textbf{if, and only if,} \quad \lambda_1 \subseteq \lambda_2 \textbf{ and } \theta_1 \sqsubseteq \theta_2.$$

### 1.6.2. Zorn's Lemma

We already pointed out that a total order is also called a linear order, and that a set equipped with a total order is often referred to as a **chain** or a **totally ordered set**. The world *chain* is however often reserved to emphasize that we are considering a subset of a partially ordered set which turns out to be totally ordered when endowed with the order relation induced by the ambient space. Formally, if $(E, \preccurlyeq)$ is a partially ordered set and $A \subseteq E$, denoted by $\preccurlyeq_A$ the restriction of $\preccurlyeq$ to $A \times A$, we say that $A$ is **a chain (in** $E$**)**, if the partially ordered set $(A, \preccurlyeq_A)$ turns out to be a totally ordered.

We say that the partially ordered set $(E, \preccurlyeq)$ is **inductive** if every chain in $E$ admits an upper bound (in $E$).

Finally, we recall that an element $m \in E$ is called **a maximal element** if, considered as a singleton $\{m\}$, it does not admit any upper bounds other than itself, i.e., if it is dominated just by itself, i.e., whenever $x \in E$ and $m \preccurlyeq x$ then necessarily $x = m$.

We admit the following axiom referred to as the ZORN's Lemma:

Max August Zorn (German: [tsɔʁn]; June 6, 1906 – March 9, 1993) was a German mathematician. He is best known for Zorn's lemma, a maximal principle in set theory that is applicable to a wide range of mathematical constructs. Zorn's lemma was first postulated by Kazimierz Kuratowski in 1922, and then independently by Zorn in 1935.

**Axiom 1.60. (Kuratowski–Zorn, 1922–1935)** *Every inductive set has at least a maximal element.*

**1.61. Remark.** Note that if $E$ is a totally ordered set then ZORN's Lemma does not add anything to the theory of sets. Indeed, to use ZORN's Lemma one has to check that $E$ is inductive; but if a totally ordered set $E$ is inductive, then $E$ (being a chain in $E$) has an upper bound in $E$. This means that $E$ has a maximum, and this is even more than the maximality guaranteed by Zorn's lemma. Thus, Zorn's lemma becomes effective only for sets that are not totally ordered.

**1.62. Remark.** Note that, roughly speaking, Zorn's Lemma permits to reduce the proof of the existence of a maximal element, in a partially ordered set $(E, \preccurlyeq)$, to something simpler: the existence of an upper bound for any of its chains. The idea behind this simplification can be partially realized by observing that a maximal element can be characterized by saying that: $m \in E$ is a maximal element if, and only if, every chain that passes through $m$ has $m$ as the maximum element. In that respect, Zorn's Lemma can be thought of as a kind of compactness result.

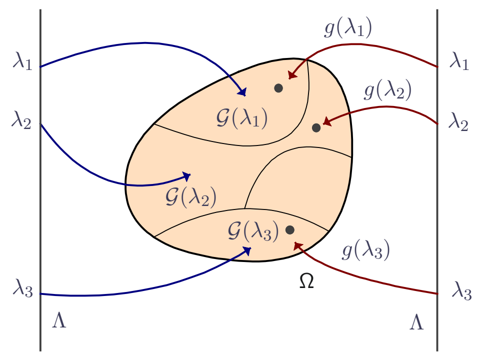



**Figure 1.8.** Axiom of choice. The function $g\colon \Lambda \to \Omega$ is a **selection map** because $g(\lambda) \in \mathcal{G}(\lambda)$ for every $\lambda \in \Lambda$.

Ernst Friedrich Ferdinand Zermelo (German: [tsɛrˈmeːlo]; 27 July 1871 – 21 May 1953) was a German logician and mathematician, whose work has major implications for the foundations of mathematics. The axiom of choice was explicitly formulated by Zermelo in 1904 and was objected to by many mathematicians. This is explained, first, by its purely existential character which makes it different from the remaining axioms of set theory and, secondly, by some of its implications which contradict intuitive common sense.

It is possible to prove that Zorn's Lemma is equivalent to the well-known axiom of choice. Namely:

**Axiom 1.63. (Zermelo, 1904)** *Let $\Lambda$ and $\Omega$ be two sets, $\wp(\Omega)$ the power set of $\Omega$. Let $\mathcal{G}$ be a map from $\Lambda$ to $\wp(\Omega)$ such that $\mathcal{G}(\lambda) \neq \emptyset$ for every $\lambda \in \Lambda$. Then, there exists a* **selection map** *$g\colon \Lambda \to \Omega$ such that $g(\lambda) \in \mathcal{G}(\lambda)$ for every $\lambda \in \Lambda$.*

### 1.6.3. The definition of generalized sequence

**1.64. Definition.** Let be $X$ **any** set. We call **generalized sequence in** $X$ (or **net in** $X$) any function defined on a directed set and with values in $X$. If $(\Lambda, \preccurlyeq)$ is a directed set, we denote a generalized sequence $x\colon \lambda \in \Lambda \to x_\lambda \in X$ by

$$\{x_\lambda\}_{\lambda \in \Lambda} \quad \textbf{or} \quad (x_\lambda)_{\lambda \in \Lambda}.$$

**Example 1.65.** Let $X$ be a set. Every ordinary sequence $x\colon (\mathbb{N}, \leqslant) \to X$ is a generalized sequence. Every function $x\colon (\mathbb{R}, \leqslant) \to X$ is a generalized sequence in $X$.

**Example 1.66.** Let $X$ be a set and $\wp(X)$ the power set of $X$ directed by inclusion (resp. reverse inclusion). To every $\emptyset \neq V \in \wp(X)$ we can associate (thanks to the **axiom of choice**) an element $x_V \in V$. In this way we obtain *two different* generalized sequence $\{x_V\}_{V \in \wp(X) \setminus \{\emptyset\}}$ in $X$, that formally differ by the chosen *direction* ($\subseteq$ or $\supseteq$).

**1.67. Remark.** The name *generalized sequence*, especially in the West, is often replaced by the term *net*. Generalized sequences (nets) play a fundamental role in topology; indeed, they are associated with the notion of Moore–Smith convergence that permits to characterize various topological properties of a topological space that cannot be caught by ordinary sequences.

From the historical point of view, was G. Birkhoff who understood the importance of directed sets in topology. He showed how to characterize topological properties by the means of the generalized notion of convergence introduced by E. H. Moore and H. L. Smith. ⋯

Birkhoff, Garrett, "Moore-Smith convergence in general topology" *Annals of Mathematics* (1937): 39-56.

E. H. Moore, and H. L. Smith, "A general theory of limits", American Journal of Mathematics, 44 (1922), 102-21

## 1.7 Filters and Filter bases

**1.68. Definition.** Let $X$ be any set and $\wp(X)$ the power set of $X$. We say that a nonempty collection $\mathcal{F}$ of subsets of $X$ is **a filter on** $X$ if it satisfies the following three conditions:

**$F_1$.** The emptyset does not belong to $\mathcal{F}$. In symbols, $\emptyset \notin \mathcal{F}$, or, $\emptyset \neq V$ for every $V \in \mathcal{F}$.

**$F_2$.** $\mathcal{F}$ is stable under finite intersections. In symbols, if $V_1, V_2 \in \mathcal{F}$ then $V_1 \cap V_2 \in \mathcal{F}$.

**$F_3$.** Every $V \in \wp(X)$ which contains an element $U \in \mathcal{F}$ also belongs to $\mathcal{F}$. In symbols, if $V \in \wp(X)$ and $V \supseteq U$ for some $U \in \mathcal{F}$, then $V \in \mathcal{F}$

Note that **$F_1$** allows for the existence of a selection map on $\mathcal{F}$. Also, note that condition **$F_3$** expresses that every filter is stable under the **superset relation**. In particular, the ambient space $X$ belongs to any filter $\mathcal{F}$ on $X$. Also, **$F_3$** implies that a filter is stable under arbitrary (in terms of cardinality) unions. Also, note that from **$F_1$** and **$F_2$** it follows that the intersection of any pair of elements $V_1$, $V_2 \in \mathcal{F}$ has nonempty intersection (and belongs to the filter).

**1.69. Definition.** We say that a nonempty collection $\mathcal{B}$ of subsets of $X$ is **a filter base of** (or **a basis for**) the filter $\mathcal{F}$ **on** $X$ if

**$FB_1$.** $\mathcal{B} \subseteq \mathcal{F}$

**$FB_2$.** Every $F \in \mathcal{F}$ contains at least an element $B \in \mathcal{B}$.

**Notation 1.70.** *Given a family of sets $\mathcal{A} \subseteq \wp(X)$ we denote by $\varpi(\mathcal{A})$[1.1] the subset of $\wp(X)$ consisting of all supersets of elements from $\mathcal{A}$. In symbols:*

$$\varpi(\mathcal{A}) = \{F \in \wp(X) :: F \supseteq A \text{ for some } A \in \mathcal{A}\}.$$

*Note that, trivially, $\varpi(\mathcal{A}) \supseteq \mathcal{A}$.*

**Example 1.71.** Let $B$ be a nonempty subset of a set $X$. The singleton $\mathcal{B} := \{B\}$ is a filter base on $X$. The filter $\varpi(\mathcal{B})$ generated by $\mathcal{B}$ consists of all subsets of $X$ containing $B$ and is called the **principal filter** generated by $B$.

**1.72. Proposition.** *Let $\mathcal{F}$ be a filter on $X$. Then $\mathcal{B} \subseteq \wp(X)$ is a filter base of $\mathcal{F}$* **if, and only if,** $\mathcal{F} = \varpi(\mathcal{B})$.

1.1. ⊕⊕⊕ The symbol $\varpi$ is named **variant pi** or **pomega**. It is a glyph variant of lower case pi sometimes used in technical contexts as though it were a lower-case omega with a macron, though historically it is simply a *cursive form of pi*, with its legs bent inward to meet. It is used as a symbol for: angular frequency of a wave in fluid dynamics (angular frequency is usually represented by $\Omega$ but this may be confused with vorticity in a fluid dynamics context); longitude of pericenter in celestial mechanics; comoving distance in cosmology; fundamental weights of a representation (to better distinguish from elements $w$ of the Weyl group, than the usual notation $\Omega$).

**Proof.** If $\mathcal{B}$ is a filter base of the filter $\mathcal{F}$, then necessarily $\varpi(\mathcal{B}) = \mathcal{F}$ where $\varpi(\mathcal{B})$ stands for the set of all those subsets of $X$ that include an element of $\mathcal{B}$.

**$\mathcal{F} \subseteq \varpi(\mathcal{B})$.** Indeed, if $F \in \mathcal{F}$ then, according to **FB$_2$**, it contains an element $B \in \mathcal{B}$ and therefore $F \in \varpi(\mathcal{B})$. Thus, $\mathcal{F} \subseteq \varpi(\mathcal{B})$.

**$\varpi(\mathcal{B}) \subseteq \mathcal{F}$.** On the other hand, if $F \in \varpi(\mathcal{B})$ then $F \supseteq B$ for some element $B \in \mathcal{B}$. But according to **FB$_1$**, $B \in \mathcal{F}$, and since $\mathcal{F}$ is stable under the superset relation and $B \subseteq F$ we have $F \in \mathcal{F}$. Hence, $\varpi(\mathcal{B}) \subseteq \mathcal{F}$.

Let us prove the sufficiency.

**FB$_1$.** Suppose $\mathcal{F} = \varpi(\mathcal{B})$ for some $\mathcal{B} \subseteq \wp(X)$. Since $\mathcal{B} \subseteq \varpi(\mathcal{B})$ we have that **FB$_1$** is satisfied.

**FB$_2$.** On the other hand, if $F \in \mathcal{F} = \varpi(\mathcal{B})$, then there exists an element $B \in \mathcal{B}$ such that $B \subseteq F$.

This concludes the proof.

**1.73. Remark.** Note that **Definition 1.69** of filter base is completely unrelated to the condition of stability under finite intersections imposed on a filter. Indeed, if $\mathcal{F}$ is a family of subsets of $X$ satisfying the properties **F$_1$** and **F$_3$** (not necessarily **F$_2$**) and if $\mathcal{B}$ satisfies **FB$_1$** and **FB$_2$**, then it still holds that $\mathcal{F} = \varpi(\mathcal{B})$. Therefore, in general, $\varpi(\mathcal{B})$ is not closed under finite intersection (if $\mathcal{F}$ is not) and, therefore, it is not a filter.

In agreement with the previous remark, it is interesting any result which guarantees that a family $\mathcal{B} \subseteq \wp(X)$ is such that $\mathcal{F} = \varpi(\mathcal{B})$ is a filter on $X$. This is the aim of the next result. Note that, given a subset $\mathcal{B}$ of $\wp(X)$, there does not exists, in general, a filter on $X$ containing $\mathcal{B}$. For example, if $A, B \subseteq X$ and $A \cap B = \emptyset$, there is no filter on $X$ containing $\{A, B\}$.

**1.74. Proposition.** *A nonempty collection $\mathcal{B}$ of subsets of $X$ is a basis for a filter $\mathcal{S}$ on $X$ if, and only if,*

**B$_1$.** *The collection $\mathcal{B}$ does not contain the emptyset among its elements;*

**B$_2$.** *The partially ordered set $(\mathcal{B}, \supseteq)$ is* **directed** (**filtered to the right**).

*The filter $\mathcal{S}$ is then $\varpi(\mathcal{B})$, and it is the smallest filter containing $\mathcal{B}$.*

**1.75. Remark.** Condition **B$_2$** means that when $\mathcal{B}$ is (partially) ordered by reverse inclusion ($(\mathcal{B}, \preccurlyeq)$ with $\preccurlyeq$ being $\supseteq$) the resulting ordered set is filtered at right, i.e., if $B_1, B_2 \in \mathcal{B}$ then there exists an upper bound $B_3$ of $\{B_1, B_2\}$. This means that $B_3$ is such that $B_1 \supseteq B_3$ and $B_2 \supseteq B_3$ or, equivalently, that $B_3 \subseteq B_1 \cap B_2$.

**Proof. The conditions are sufficient.** Set $\mathcal{S} := \varpi(\mathcal{B})$. Then, $\mathcal{S}$ is a filter on $X$. Indeed, $\mathcal{S}$ does not contain the emptyset because $\mathcal{B}$ does not. Moreover, if $V \in \wp(X)$ contains an element of $\varpi(\mathcal{B})$ then clearly $V \in \varpi(\mathcal{B})$. Finally, we show that $\mathcal{S}$ is stable under finite intersections. For any $A_1, A_2 \in \mathcal{S}$ there exist $B_1, B_2 \in \mathcal{B}$ such that $A_1 \supseteq B_1$, $A_2 \supseteq B_2$. Since $(\mathcal{B}, \supseteq)$ is a directed set, there exists $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$. Thus, $B_3 \subseteq A_1 \cap A_2$ and this implies $A_1 \cap A_2 \in \varpi(\mathcal{B}) = \mathcal{S}$.

**The conditions are necessary.** Condition **B$_1$** is trivially necessary (because the first property of being a filter basis for $\mathcal{S}$ is $\mathcal{B} \subseteq \mathcal{S}$, i.e., that if $B \in \mathcal{B}$ then $B \in \mathcal{S}$). Let us prove that $(\mathcal{B}, \supseteq)$ is directed. Let $B_1, B_2 \in \mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{S}$ we have $B_1, B_2 \in \mathcal{S}$. But $\mathcal{S}$ is a filter and therefore $F_3 := B_1 \cap B_2$ is in

$\mathcal{S}$. Also, as $\mathcal{B}$ is a filter base of $\mathcal{S}$ there exists $\mathcal{B} \ni B_3 \subseteq F_3$. Therefore, given $B_1, B_2 \in \mathcal{B}$ (passing through the filter $\mathcal{S}$) we were able to pick up an element $B_3 \in B$ such that $B_3 \subseteq B_1 \cap B_2$.

**1.76. Remark.** Note the duality character of the results stated in **Proposition 1.72** and in **Proposition 1.74**. The first proposition assumes that we already have a filter $\mathcal{F}$ at our disposal, and characterizes all possible subsets of $\mathcal{F}$ which turn out to be a filter base of $\mathcal{F}$. On the other hand, **Proposition 1.74** investigates under which conditions on a subset $\mathcal{B}$ of $\wp(X)$ the construction $\mathcal{F} := \varpi(\mathcal{B})$ produces a filter on $X$.

Before stating the next result, let us make a simple observation. If $\mathcal{B}_1, \mathcal{B}_2$ are two subsets of $\wp(X)$ such that $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\forall B_2 \in \mathcal{B}_2 \, \exists B_1 \in \mathcal{B}_1 :: B_1 \subseteq B_2$, then

$$\varpi(\mathcal{B}_1) = \varpi(\mathcal{B}_2). \tag{1.24}$$

Indeed, the inclusion $\varpi(\mathcal{B}_1) \subseteq \varpi(\mathcal{B}_2)$ is trivial. On the other hand, if $F_2 \in \varpi(\mathcal{B}_2)$ then $F_2 \supseteq B_2$ for some $B_2 \in \mathcal{B}_2$, and $B_2 \supseteq B_1$ for some $B_1 \in \mathcal{B}_1$. Hence, $F_2 \supseteq B_1$, i.e., $F_2 \in \varpi(\mathcal{B}_1)$. After that, from **Proposition 1.72** we immediately get the following result.

**1.77. Corollary.** *Let $\mathcal{B}_2$ be a filter base of the filter $\mathcal{F}$ on $X$. Suppose that $\mathcal{B}_1 \subseteq \wp(X)$ is such that $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\forall B_2 \in \mathcal{B}_2 \, \exists B_1 \in \mathcal{B}_1 :: B_1 \subseteq B_2$. Then $\mathcal{B}_1$ is still a filter base of the filter $\mathcal{F}$ on $X$.*

More generally, the following result holds.

**1.78. Corollary.** *Let $\mathcal{B}_1 \subseteq \wp(X)$ be a filter base of the filter $\mathcal{F}_1 := \varpi(\mathcal{B}_1)$ on $X$ and let $\mathcal{B}_2 \subseteq \wp(X)$ be a filter base of the filter $\mathcal{F}_2 := \varpi(\mathcal{B}_2)$ on $X$. Suppose that*

$$\forall B_2 \in \mathcal{B}_2 \, \exists B_1 \in \mathcal{B}_1 :: B_1 \subseteq B_2 \quad \textbf{and} \quad \forall B_1 \in \mathcal{B}_1 \, \exists B_2 \in \mathcal{B}_2 :: B_2 \subseteq B_1. \tag{1.25}$$

*Then, $\varpi(\mathcal{B}_1) = \varpi(\mathcal{B}_2)$.*

The previous result can be rephrased in a more suggestive way. To this end, let us introduce the following definition.

**1.79. Definition.** Let $\mathcal{B}_1 \subseteq \wp(X)$ be a filter base of the filter $\mathcal{F}_1 := \varpi(\mathcal{B}_1)$ on $X$, and let $\mathcal{B}_2 \subseteq \wp(X)$ be a filter base of the filter $\mathcal{F}_2 := \varpi(\mathcal{B}_2)$ on $X$. We say that the filter base $\mathcal{B}_2$ is finer than $\mathcal{B}_1$ if

$$\varpi(\mathcal{B}_1) \subseteq \varpi(\mathcal{B}_2).$$

We say that the filter bases $\mathcal{B}_1, \mathcal{B}_2$ are equivalent when $\varpi(\mathcal{B}_1) = \varpi(\mathcal{B}_2)$.

It is simple to show that $\mathcal{B}_2$ is finer than $\mathcal{B}_1$ if, and only if the second relation in (1.25) holds, that is, if, and only, if

$$\forall B_1 \in \mathcal{B}_1 \, \exists B_2 \in \mathcal{B}_2 :: B_2 \subseteq B_1. \tag{1.26}$$

Therefore, relation (1.25) gives necessary and sufficient conditions for the two filter bases $\mathcal{B}_1$ and $\mathcal{B}_2$ to be equivalent.

# 2

## Some topological concepts

### 2.1 | Topological spaces (via neighborhoods)

For our purposes, it is convenient to introduce topological spaces through the axiomatization introduced by Felix Hausdorff in 1914 and based on the notion of a filter of neighborhoods of a point.

Felix Hausdorff (November 8, 1868 – January 26, 1942) was a German mathematician. He is considered to be one of the founders of modern topology.

Let $X$ be a non-empty set. Let $\mathcal{V}$ be a function assigning to each $x \in X$ a nonempty collection $\mathcal{V}(x) \subseteq \wp(X)$ of subsets of $X$. We say that $(X, \mathcal{V})$ is a **topological space** in the sense of Hausdorff (or that $\mathcal{V}$ defines a **topology on** $X$) if, for every $x \in X$, $\mathcal{V}(x)$ is the **filter of neighborhoods of** $x$, that is, if the following *axioms* are satisfied:

**$H_1$.** For any $x \in X$ the family $\mathcal{V}(x)$ is a filter on $X$: ▶ The emptyset does not belong to $\mathcal{V}(x)$; ▶ $\mathcal{V}(x)$ is stable under finite intersection. ▶ Every $V \in \wp(X)$ containing an element $U \in \mathcal{V}(x)$ also belongs to $\mathcal{V}(x)$.

**$H_2$.** The point $x \in X$ belongs to every element $V \in \mathcal{V}(x)$. In other words, $\{x\} \subseteq \bigcap_{V \in \mathcal{V}(x)} V$.

**$H_3$.** Given any $V \in \mathcal{V}(x)$, there exists a $W \in \mathcal{V}(x)$ such that for any $y \in W$ one has $V \in \mathcal{V}(y)$. In formulae:

$$\forall V \in \mathcal{V}(x)\, \exists W \in \mathcal{V}(x) :: V \in \mathcal{V}(y) \quad \textbf{for any} \quad y \in W.$$

Note that, due to **$H_2$**, one necessarily has $y \in V$ for every $y \in W$. Hence, $W(x) \subseteq V(x)$.

The elements of $\mathcal{V}(x)$ will be called **neighborhoods** of $x$. The function $\mathcal{V}$ is referred to as a **neighborhood topology** on $X$. If $(X, \mathcal{V})$ is a **topological space**, we refer to $X$ as its **carrier set**. The elements of $X$ are called **points** of $X$. The relation $V \in \mathcal{V}(x)$ reads as «$V$ is a neighborhood of the point $x \in X$». Also, if $W \subseteq X$ and $V \in \mathcal{V}(y)$ for every $y \in W$, then we say that $V$ **is a neighborhood of** $W$. In other words, if $V$ is a neighborhood of every point of a set $W$, we say that $V$ is a neighborhood of $W$.

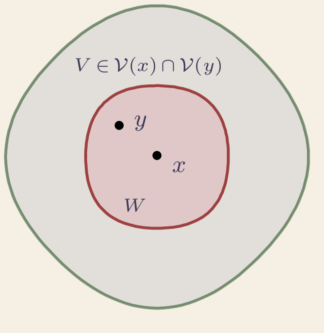



**Figure 2.1.** **Axiom $H_3$.** Given any $V \in \mathcal{V}(x)$, there exists a $W \in \mathcal{V}(x)$ such that for any $y \in W$ one has $V \in \mathcal{V}(y)$. Note that, due to **$H_2$**, one necessarily has $y \in V$ for every $y \in W$ and therefore $W \subseteq V$. Also, $V \in \mathcal{V}(y) \cap \mathcal{V}(x)$ for any $y \in W$.

**2.1. Remark.** Note that, if $V$ is a neighborhood of $W$ then necessarily $W \subseteq V$ (due to **$H_2$**). Axiom **$H_3$** can then be stated in the following equivalent forms: «every neighborhood $V$ of $x \in X$ is a neighborhood for some smaller neighborhood $W \in \mathcal{V}(x)$», or «any neighborhood $V$ of $x \in X$ includes a (smaller) neighborhood $W$ of $x$ such that $V$ is a neighborhood of each point of $W$». We soon introduce the concept of interior of a set that allows to reformulate **$H_3$** under the equivalent form: «every neighborhood of $x$ has nonempty interior».

Given a neighborhood topology $\mathcal{V}$ on $X$, for every $x \in X$ the collection $\mathcal{V}(x)$, being the filter of neighborhoods of $x$, is, in particular (by **$H_1$**), a filter. Every filter base of $\mathcal{V}(x)$ is called a **basis of neighborhoods** of $x$ (or **a fundamental system of neighborhoods of** $x$). We shall usually denote a fundamental system of neighborhoods of $x$ by $\mathcal{B}(x)$. One then obtains $\mathcal{V}(x)$ by considering the family $\varpi(\mathcal{B}(x))$ of all supersets of elements in $\mathcal{B}(x)$. Therefore, the neighborhood topology on $X$, can also be defined by assigning a fundamental system of neighborhoods $\mathcal{B}(x)$ to every $x$ in $X$.

If **every** point of $X$ has a basis (of neighborhoods) consisting of countably many neighborhoods, then we say that $X$ is **a first-countable space** (or that it satisfies the **first axiom of countability**, or that it has a **countable local basis**).

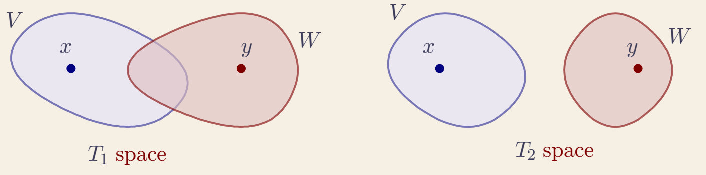



**Figure 2.2.** LEFT. If $x$ and $y$ are two distinct elements in $X$, there exist $V \in \mathcal{V}(x)$ and $W \in \mathcal{V}(y)$ such that $y \notin V$ and $x \notin W$. RIGHT. If $x$ and $y$ are two distinct elements in $X$, there exist $V \in \mathcal{V}(x)$ and $W \in \mathcal{V}(y)$ such that $V \cap W = \emptyset$

A topological space $X$ is said to be **(Hausdorff) separated** or a **Hausdorff space** when it satisfies the following **Hausdorff separation axiom** (also known as the **$T_2$-separation axiom**):

**Axiom 2.2. ($T_2$ Hausdorff separation axiom)** *If $x$ and $y$ are two distinct elements in $X$, there exist $V \in \mathcal{V}(x)$ and $W \in \mathcal{V}(y)$ such that $V \cap W = \emptyset$.*

Also, let us recall that a topological space $(X, \mathcal{V})$ is said to be a $T_1$ space when it satisfies the following separation axiom introduced by Fréchet:

**Axiom 2.3. ($T_1$ Fréchet separation axiom)** *If $x$ and $y$ are two distinct elements in $X$, there exist $V \in \mathcal{V}(x)$ and $W \in \mathcal{V}(y)$ such that $y \notin V$ and $x \notin W$. Note that, for a $T_1$ space, $V$ and $W$ are not required to be disjoint.*

**2.4. Remark.** Clearly, every $T_2$ space is also a $T_1$ space. Moreover, any metric space $(X, d)$ is $T_2$ and, therefore, $T_1$. One may wonder why we should extrapolate, mimicking what happens in metric space, such a separation property as $T_1$. The reason is that a $T_1$ space allows for an argument omnipresent in analysis, mainly that if $x, y \in X$ and $d(x, y) < \varepsilon$ for every $\varepsilon > 0$ then necessarily $x = y$ (even more concretely, if $x \in \mathbb{R}$ and $|x| < \varepsilon$ for every $\varepsilon$ then $x = 0$). In fact, the equivalent statement in a $T_1$ space reads as follows.

**2.5. Proposition.** *Suppose that* $(X, \mathcal{V})$ *is a* $T_1$ *space and let* $x, y \in X$. *If* $y \in V$ *for every* $V \in \mathcal{V}(x)$, *or if* $x \in W$ *for every* $W \in \mathcal{V}(y)$, *then necessarily* $y = x$.

The proof is straightforward. Suppose that $y \in V$ for every $V \in \mathcal{V}(x)$ but $y \neq x$. Since $X$ is $T_1$ there exists $U \in \mathcal{V}(x)$ such that $y \notin U$ and this contradicts the assumption. Therefore, it is necessarily $y = x$.

**Example 2.6. Metric topology.** Let $(X, d)$ be a metric space. We can define a topology on $X$ by taking as basis of neighborhoods of $x \in X$ the set of all open balls $\{B_\circ(x, 1/n) :: x \in X, n \in \mathbb{N}^*\}$ or the set of all closed balls $\{B_\bullet(x, 1/n) :: x \in X, n \in \mathbb{N}^*\}$. Thus, the metric topology satisfies the first axiom of countability. Also, it is easy to show that the metric topology is (Hausdorff) separated.

### 2.1.1. Interior, closure (or adhérence)

Let $(X, \mathcal{V})$ be a topological space and $A, B$ subsets of $X$.

**2.7. Definition.** We say that a point $x \in X$ **is interior to** $A$ if $A$ is a neighborhood of $x$, i.e., if $A \in \mathcal{V}(x)$. The set of points in $X$ which are interior to $A$ is called **the interior** of $A$ and denoted by $A^\circ$ or by $\operatorname{int}_X(A)$. Formally,

$$A^\circ := \{y \in X :: A \in \mathcal{V}(y)\}. \tag{2.1}$$

By **Axiom $\mathbf{H_2}$** of the filter of neighborhoods (i.e., each neighborhood of a point contains that point), we have $A^\circ \subseteq A$. Therefore, (2.1) is equivalent to

$$A^\circ := \{y \in A :: A \in \mathcal{V}(y)\}. \tag{2.2}$$

A set $A \subseteq X$ is called **open** when $A^\circ \supseteq A$ (and in this case we have $A = A^\circ$).

Formally, $A$ is open, if and only if, $A \in \mathcal{V}(a)$ for every $a \in A$, i.e., when $A$ is a neighborhood of each of its points. In particular, $X$ is an open set. Also, $\emptyset^\circ := \{x \in X :: \emptyset \in \mathcal{V}(x)\} \equiv \emptyset$ and, therefore, $\emptyset$ is open.

**2.8. Proposition.** *Let* $(X, \mathcal{V})$ *be a topological space,* $x \in X$. *The following properties hold:*

***i.*** *Every neighborhood of* $x$ *has a further neighborhood included in the interior of it. In particular, every neighborhood has nonempty interior. It follows that if* $V \in \mathcal{V}(x)$ *then* $V^\circ$ *is an open neighborhood of* $x$. *In particular,* $V^\circ \neq \emptyset$.

***ii.*** *The interior operator is idempotent. In other words, for any* $A \subseteq X$ *one has* $A^{\circ\circ} = A^\circ$. *This is equivalent to say that* $A^\circ$ *is an open set.*

*In particular, every neighborhood of* $x \in X$ *contains an open neighborhood (namely, its interior).*

**Proof.** ***i.*** Let $V \in \mathcal{V}(x)$ be a neighborhood of the point $x \in X$. By definition, cf. (2.2), we have

$$V^\circ := \{y \in V :: V \in \mathcal{V}(y)\}.$$

By the **Axiom $\mathbf{H_3}$**, we know that there exists $W \in \mathcal{V}(x)$ such that ($\emptyset \neq W \subseteq V$ and) $V \in \mathcal{V}(y)$ for any $y \in W$. But this means that $W \subseteq V^\circ$. Since $V^\circ$ contains a neighborhood of $x$ (namely $W$), and $\mathcal{V}(x)$ is a filter, necessarily $V^\circ \in \mathcal{V}(x)$. In particular, $x \in V^\circ \neq \emptyset$.

***ii.*** It is sufficient to prove that $A^\circ \subseteq A^{\circ\circ}$. Note that, by definition, cf. (2.2), we have

$$A^{\circ\circ} := \{y \in A^\circ :: A^\circ \in \mathcal{V}(y)\}.$$

Now, let $y \in A^\circ$. By definition, $A \in \mathcal{V}(y)$. Also, by ***i.***, we have $A^\circ \in \mathcal{V}(y)$. Therefore $y \in A^{\circ\circ}$. ■■■■

**2.9. Definition.** We say that $x \in X$ is an **adherént point** (or a **closure point**) of $A$, if every neighborhood of $x$ meets $A$, that is, if $A \cap V \neq \emptyset$ for every $V \in \mathcal{V}(x)$. The set of all points that adhere to $A$ is called the **adhérence** (or the **closure**) of $A$ and is denoted by $\bar{A}$ or by $\mathrm{cl}_X(A)$. Formally,

$$\bar{A} := \{x \in X :: A \cap V \neq \emptyset \text{ for every } V \in \mathcal{V}(x)\}.$$

By the **Axiom $\mathbf{H_2}$** of the filter of neighborhoods we have $A \subseteq \bar{A}$ (as for every $x \in A$ we have $\{x\} \subseteq A \cap V$ for every $V \in \mathcal{V}(x)$). A set $A$ is called **closed** if $A \supseteq \bar{A}$ (and in this case we have $A = \bar{A}$).

**2.10. Definition.** We say that a set $A$ **is dense** in $B$ when $\bar{A} \supseteq B$.

The elementary properties concerning the open and closed sets are collected in the next result.

**2.11. Proposition.** *Let $(X, \mathcal{V})$ be a topological space. The following properties hold:*

**Properties of open sets.**

- *The family of open sets contains $X$ and $\emptyset$. Moreover, it is stable under finite intersections and arbitrary unions.*
- *Every point $x$ of a topological space $X$ has a basis of neighborhoods composed by open sets. Indeed, every neighborhood $V$ of $x$ contains the open neighborhood of $x$ given by $V^\circ$.*

**Properties of closed sets.**

*The complement of an open set is a closed set. Therefore, the properties of closed sets can be derived by (Boolean) duality from the corresponding properties of the open sets. In particular, the family of closed sets contains $X$ and $\emptyset$. Moreover, it is stable under finite unions and arbitrary intersections. If $X$ is a Hausdorff space, every singleton is a closed set.*

In **Proposition 2.11** we recalled that every point of a topological space has a basis of neighborhoods consisting of open sets. However, it is not always the case that each point of a topological space has a basis of neighborhoods consisting of closed sets. This observation justifies the next definition.

**2.12. Definition.** We say that a topological space is **regular** if **each of its points** admits a basis of neighborhoods of consisting of closed sets (actually, of regular open sets).

Recall that an open set $R$ is called **regular** if, and only if $R = R^{-\circ}$.

○

### 2.1.2. The induced (subspace) topology

Let $X$ be a topological space and $M$ a subset of $X$.

**2.13. Definition.** We say that $M$ is endowed with **the topology induced by** $(X, \mathcal{V}_X)$ when the filter of neighborhoods of the generic point $m \in M$, that we denote by $\mathcal{V}_M(m)$, is defined as

$$\mathcal{V}_M(m) := \{V_M \subseteq M :: V_M = V \cap M, V \in \mathcal{V}_X(m)\}.$$

We then say that $(M, \mathcal{V}_M)$ is a (topological) subspace of $X$. For any $m \in M$, the collection $\{V \cap M\}_{V \in \mathcal{V}_X(m)}$ is also referred to as the trace of the filter $\mathcal{V}_X(m)$ on $M$.

Is it simple to show that, for every $m \in M$, $\mathcal{V}_M(m)$ is a filter of neighborhoods of $m$.

**2.14. Remark.** Given a filter $\mathcal{F}$ on a set $X$, and $A$ a subset of $X$, the trace of $\mathcal{F}$ on $A$ is defined as the family of sets $\mathcal{F}_A := \{F \cap A :: F \in \mathcal{F}\}$. Sometimes also the notation $\mathcal{F}|A$ is used. It is simple to show that the trace of $\mathcal{F}$ on $A$ is a filter if, and only if, each set of $\mathcal{F}$ meets $A$ (i.e., has nonempty intersection with $A$). In this setting, we can say that the topology induced by $(X, \mathcal{V}_X)$ on a subset $M \subseteq X$ is the function that sends each $m \in M$ to the trace of $\mathcal{V}_X(m)$ on $M$.

### 2.1.3. Comparison of topologies

The set of all possible topologies on a set $X$, and by this we mean the set of all possible functions $\mathcal{V}: x \in X \mapsto \mathcal{V}(x)$ satisfying $\mathbf{H_1}$, $\mathbf{H_2}$, and $\mathbf{H_3}$, can be naturally structured into a partially ordered set. This order relation can be used to compare different topologies defined on the **same carrier set**. More precisely, given two topological spaces $(X, \mathcal{V}_1)$, $(X, \mathcal{V}_2)$, *having the same carrier set* $X$, we say that $\mathcal{V}_1$ is **finer** (or **stronger**) than $\mathcal{V}_2$ (or that $\mathcal{V}_2$ is **weaker** than $\mathcal{V}_1$), and in this case we write $\mathcal{V}_1 \supseteq \mathcal{V}_2$, if for every $x \in X$ one has $\mathcal{V}_1(x) \supseteq \mathcal{V}_2(x)$.

Given two topological spaces $(X_1, \mathcal{V}_1)$, $(X_2, \mathcal{V}_2)$, we write $X_1 \hookrightarrow X_2$ whenever $X_1 \subseteq X_2$ and the topology $\mathcal{V}_1$ is finer than the topology induced by $\mathcal{V}_2$ on $X_1$. Formally,

$$X_1 \hookrightarrow X_2 \quad \text{if, and only if,} \quad X_1 \subseteq X_2 \text{ and } \{V \cap X_1\}_{V \in \mathcal{V}_2(x)} \subseteq \mathcal{V}_1(x) \; \forall x \in X_1.$$

○

### 2.1.4. The product topology

**2.15. Definition.** Let $X$ and $Y$ be two topological spaces. We call **topological product space of** $X$ **and** $Y$, the cartesian product $X \times Y$ endowed with the following topology: given any $(x, y) \in X \times Y$ and two bases of neighborhoods $\mathcal{B}(x)$ and $\mathcal{B}(y)$ of $x$ and $y$, we build a basis of neighborhoods of $(x, y)$ by setting

$$\mathcal{B}(x, y) := \mathcal{B}(x) \otimes \mathcal{B}(y)$$

where $\mathcal{B}(x) \otimes \mathcal{B}(y) := \{B_1 \times B_2 :: (B_1, B_2) \in \mathcal{B}(x) \times \mathcal{B}(y)\}$.

It is simple to show that $\mathcal{B}(x, y)$ satisfies the criteria of **Proposition 1.74** and, therefore, that the condition $\mathbf{H_1}$ is satisfied. Thus, $\mathcal{B}(x, y)$ is a filter base for the filter $\varpi(\mathcal{B}(x, y))$ on $X$. Moreover, one can easily check that $\mathcal{V}(x, y) := \varpi(\mathcal{B}(x, y))$ also satisfies conditions $\mathbf{H_2}$ and $\mathbf{H_3}$ and, therefore, the map $(x, y) \in X \times Y \mapsto \mathcal{V}(x, y)$ defines a neighborhood topology on $X \times Y$.

**2.16. Proposition.** *Let $Z = X \times Y$ be the topological product of the topological spaces $X$ and $Y$. For any $A \times B \subseteq X \times Y$ we have $\overline{A \times B} = \bar{A} \times \bar{B}$. In other words, the closure of any cartesian product included in the product space coincides with the product of their closures.*

### 2.1.5. Limit of a generalized sequence

**2.17. Definition.** Let $(X, \mathcal{V})$ be a topological space and $(\Lambda, \preccurlyeq)$ a directed set. We say that the generalized sequence $(x_\lambda)_{\lambda \in \Lambda}$ converges to a point $x \in X$ (not necessarily unique) if

$$\forall V \in \mathcal{V}(x), \; \exists \lambda_0 \in \Lambda \;::\; x_\lambda \in V \text{ when } \lambda \succcurlyeq \lambda_0. \tag{2.3}$$

In this case, **we write**

$$x \in X\text{-}\lim_{\Lambda} x_\lambda, \tag{2.4}$$

or, very often, $x \in \lim_\Lambda x_\lambda$ when no confusion may arise.

We remark that, in general, $\lim_\Lambda x_\lambda$ can reduce to the emptyset. Also, we emphasize that, here, the symbol $\succcurlyeq$ stands for the inverse relation of $\preccurlyeq$, so that $\lambda \succcurlyeq \lambda_0$ stand for $\lambda_0 \preccurlyeq \lambda$.

**Terminology.** Given a generalized sequence $(x_\lambda)_{\lambda\in\Lambda}$ in $X$, we say that $(x_\lambda)_{\lambda\in\Lambda}$ **eventually** satisfies a prescribed property if all terms beyond some $\lambda_0 \in \Lambda$ have that property. For example, if $(\Lambda, \preccurlyeq)$ has $\lambda^* \in \Lambda$ has maximum element, then any map $x: \Lambda \to X$ is eventually constant. Indeed, in this case, the set $\{\lambda \in \Lambda :: \lambda \succcurlyeq \lambda^*\}$ reduces to the singleton $\{\lambda^*\}$ so that, with $\lambda_0 := \lambda^*$, we get that $x_\lambda = x^*$ whenever $\lambda \succcurlyeq \lambda_0$. After that, we can say that $(x_\lambda)_{\lambda\in\Lambda}$ converges to a point $x \in X$ if for every $V \in \mathcal{V}(x)$, $(x_\lambda)_{\lambda\in\Lambda}$ is *eventually* in $V$.

**2.18. Remark.** Note that we get an equivalent definition if we replace the filter of neighborhoods $\mathcal{V}(x)$ by any basis $\mathcal{B}(x)$ of the filter of neighborhoods.

**2.19. Remark.** Note that the definition of a limit of a generalized sequence makes sense also when $(\Lambda, \preccurlyeq)$ is only a partially ordered set. However, in the context of topological spaces, the notion is useful and interesting under the stronger condition that $(\Lambda, \preccurlyeq)$ is a directed set. In fact, although certain results still hold when $(\Lambda, \preccurlyeq)$ is a partially ordered set, to have, e.g., uniqueness of the limit in a Hausdorff separated topological space one needs $(\Lambda, \preccurlyeq)$ to be directed.

**Example 2.20.** Let $\mathcal{B}(x_0)$ be a basis of neighborhoods of $x_0$. The partially ordered set $(\mathcal{B}(x_0), \supseteq)$ is directed set because for every $B_1, B_2 \in \mathcal{B}(x_0)$ there exists $B_3 \in \mathcal{B}(x_0)$ such that $B_3 \subseteq B_1 \cap B_2$. In particular, if $\mathcal{V}(x_0)$ is the filter of neighborhoods of $x_0$, then $(\mathcal{V}(x_0), \supseteq)$ is directed.

But then, for every $B \in \mathcal{B}(x_0)$ we can (arbitrarily) select an element $x_B \in B$ (thanks to the axiom of choice) to get a generalized sequence $(x_B)_{B\in\mathcal{B}(x_0)}$ that we claim converges to $x_0$. To see this, it is useful to specialize condition (2.3) to the present context. By definition, the generalized sequence $(x_B)_{B\in\mathcal{B}(x_0)}$ converges to $x_0$ if, and only if,

Note that as the directed set is $(\mathcal{B}(x_0), \supseteq)$, the order relation $\preccurlyeq$ is now $\supseteq$. Therefore the condition «when $\lambda \succcurlyeq \lambda_0$» reads now as «when $\lambda \subseteq \lambda_0$».

$$\forall V \in \mathcal{V}(x), \quad \exists B_0 \in \mathcal{B}(x_0) :: x_B \in V \text{ whenever } \mathcal{B}(x_0) \ni B \subseteq B_0.$$

Now, for any $V \in \mathcal{V}(x_0)$ there exists $B_0 \in \mathcal{B}(x_0)$ such that $B_0 \subseteq V$. Hence, if $B \subseteq B_0$ then $x_B \in B \subseteq B_0 \subseteq V$. Therefore, $x_0 \in \lim_{\mathcal{B}(x_0)} x_B$. ⋯

**Example 2.21. Indiscrete topology.** In general, the limit of a generalized sequence can is not unique. For example, we can define a neighborhood topology on a set $X$ considering the constant function $\mathcal{V}: x \mapsto \{X\} \subseteq \wp(X)$. Now, recall that a subset $A \subseteq X$ is open if it is a neighborhood of each of its points. Since the only neighborhood is $X$, the only two open sets are $X$ and $\emptyset$. The topology so defined is called the indiscrete topology.

It is easily seen that if $x_0 \in X$, $\mathcal{B}(x_0) := \{X\}$ is a basis of neighborhoods of $x_0$ and $(\mathcal{B}(x_0), \supseteq)$ is a directed set. If we choose a $x_B \in X$ we get that the generalized sequence $(x_B)_{B\in\mathcal{B}(x_0)}$ converges to $x_0$ but also to any other point of $X$. Indeed, the generalized sequence $(x_B)_{B\in\mathcal{B}(x_0)}$ converges to $x \in X$ if, and only if,

$$\forall V \in \mathcal{V}(x), \quad \exists B_0 \in \mathcal{B}(x_0) :: x_B \in V \text{ whenever } \mathcal{B}(x_0) \ni B \subseteq B_0.$$

Therefore, taking into account that $\mathcal{V}(x)=\mathcal{B}(x)=\{X\}$ for any $x\in X$, the relation $x_B\in X$ holds for any $V\in\mathcal{V}(x)$. Thus, $\lim_{\mathcal{B}(x_0)} x_B\equiv X$. ⋯

**Example 2.22. Discrete topology.** We can define a neighborhood topology on a set $X$ considering the family of filters generated by $\mathcal{B}: x\mapsto\{x\}\subseteq\wp(X)$, that is the function $\mathcal{V}: x\mapsto\varpi(\{x\})$. Now, a subset $A\subseteq X$ is open if it is a neighborhood of each of its points. Since every subset of $X$ passing through $x\in X$ is a neighborhood of $X$, every subset of $X$ is open. The topology so defined is called the discrete topology.

Given any generalized sequence $(x_\lambda)_{\lambda\in\Lambda}$, a point $x_0\in X$ belongs to $\lim_\Lambda x_\lambda$ if, and only if,

$$\forall V\in\mathcal{V}(x_0),\quad \exists\lambda_0\in\Lambda :: x_\lambda\in V \text{ whenever } \lambda\geqslant\lambda_0.$$

Now, taking $V=\{x_0\}$, we get that if $x_0\in\lim_\Lambda x_\lambda$ then $(x_\lambda)_{\lambda\in\Lambda}$ is eventually constant and equal to $x_0$. Therefore, if $(x_\lambda)_{\lambda\in\Lambda}$ is not eventually constant, we have $\lim_\Lambda x_\lambda=\emptyset$. ⋯

**2.23. Proposition.** *Let $M\subseteq X$. Let $(m_\lambda)_{\lambda\in\Lambda}$ be a generalized sequence in $M$ and $m\in M$. Then $(m_\lambda)_{\lambda\in\Lambda}$ converges to $m$ for the topology on $M$ induced by $X$, if, and only if, $(m_\lambda)$ converges to $m$ for the topology of $X$. In other terms:*

$$M\text{-}\lim_\Lambda m_\lambda = M\cap\left(X\text{-}\lim_\Lambda m_\lambda\right).$$

*Here, we have denoted by $M$-lim the limit operator relative to the subset $M$ endowed with the subspace topology induced by $X$ on $M$.*

**Proof.** Suppose $m\in M\text{-}\lim_\Lambda m_\lambda$. By definition, for every neighborhood $V_M\in\mathcal{V}_M(m)$ there exists $\lambda_0\in\Lambda$ such that $m_\lambda\in V_M$ for any $\lambda\succcurlyeq\lambda_0$. Observe that, for every $V_X\in\mathcal{V}_X(m)$, $V_M:=V_X\cap M$ is an element of $\mathcal{V}_M(m)$ and therefore, there exists $\lambda_0\in\Lambda_0$ such that for any $\lambda\succcurlyeq\lambda_0$, we have $m_\lambda\in V_X\cap M\subseteq V$. Hence, $m\in X\text{-}\lim_\Lambda m_\lambda$.

On the other hand, if $m\in M\cap(X\text{-}\lim_\Lambda m_\lambda)$, then for every neighborhood $V_X\in\mathcal{V}_X(m)$ there exists $\lambda_0\in\Lambda$ such that $m_\lambda\in V_X$ for any $\lambda\succcurlyeq\lambda_0$. In particular, for every $V_M:=V_X\cap M$, as $(m_\lambda)_{\lambda\in\Lambda}$ takes values in $M$, there exists $\lambda_0\in\Lambda$ such that $m_\lambda\in V_X\cap M$ for any $\lambda\succcurlyeq\lambda_0$. This completes the proof. ■

**2.24. Proposition.** *Let $Z$ be the topological product space of the topological spaces $X$ and $Y$. Let $(z_\lambda=(x_\lambda,y_\lambda))_{\lambda\in\Lambda}$ be a generalized sequence in $Z$ and $z=(x,y)\in Z$. Then $(z_\lambda)_{\lambda\in\Lambda}$ converges to $z$ if, and only if, $(x_\lambda)_{\lambda\in\Lambda}$ converges to $x$ and $(y_\lambda)_{\lambda\in\Lambda}$ converges to $y$.*

### 2.1.6. Uniqueness of the limit of a generalized sequence

**2.25. Remark.** Let $(x_\lambda)_{\lambda\in\Lambda}$ be a generalized sequence, and consider a finite number of predicates $P_1,P_2,\ldots,P_n$. If $(x_\lambda)_{\lambda\in\Lambda}$ eventually satisfies $P_1$, eventually satisfies $P_2$, …, and eventually satisfies $P_n$, then, since $\Lambda$ is *directed* we have that $(x_\lambda)_{\lambda\in\Lambda}$ eventually satisfies $P_1\wedge P_2\wedge\ldots\wedge P_n$. Here, the symbol $\wedge$ stands for the logical and operator.

**2.26. Proposition.** *In a (Hausdorff) separated topological space the limit of a generalized sequence, whenever it exists, is unique.*

**Proof.** Let $(X,\mathcal{V})$ be a topological space, and let $x_1,x_2\in X$ be limits (not necessarily distinct) of the same generalized sequence $(x_\lambda)_{\lambda\in\Lambda}$. In other words, suppose that

$$\{x_1,x_2\}\subseteq\lim_{\Lambda}x_\lambda.$$

Let $V_1\in\mathcal{V}(x_1)$ and $V_2\in\mathcal{V}(x_2)$. Since $x_\lambda\to x_1$ we have that $(x_\lambda)_{\lambda\in\Lambda}$ is eventually in $V_1$. Also, since $x_\lambda\to x_2$ we have that $(x_\lambda)_{\lambda\in\Lambda}$ is eventually in $V_2$. Since the set $\Lambda$ is **directed**, by **Remark 2.25** we infer that $x_\lambda$ is eventually in $V_1\cap V_2$. Overall, we proved that $V_1\cap V_2\neq\emptyset$ for every $V_1\in\mathcal{V}(x_1)$ and $V_2\in\mathcal{V}(x_2)$. But this implies that necessarily $x_1=x_2$ if $X$ is Hausdorff separated.

**2.27. Remark.** Actually, the content of **Proposition 2.26** is an "if, and only if". The only if part follows from the observation that if $X$ is not Hausdorff then there exit two points $x_1\neq x_2$ such that $V_1\cap V_2\neq\emptyset$ for every $V_1\in\mathcal{V}(x_1)$ and $V_2\in\mathcal{V}(x_2)$. Selecting an element $(x_B)$ from every neighborhood of the type $B=V_1\cap V_2$ with $V_1\in\mathcal{V}(x_1)$, $V_2\in\mathcal{V}(x_2)$, we build a generalized sequence which converges both to $x_1$ and $x_2$.

### 2.1.7. Characterization of the adherence (closure)

**2.28. Proposition.** *Let $A$ be a subset of a topological space $X$. Then, $x\in\bar{A}$ if, and only if, there exists a generalized sequence of points* **in** *$A$ which converges (for the topology of $X$) to $x$. If $X$ has a* **countable basis of neighborhoods** *(that is, when $X$ satisfies the first axiom of countability) one can replace* **generalized sequences** *with* **ordinary sequences**.

**2.29. Remark.** Note that the closure has to be tested with generalized sequences that take values in $A$. Otherwise, every element of the space would belong to the closure: just consider, for $x\in X$, the constant generalized sequence $\lambda\in\Lambda\mapsto x\in X$ converging to $x$.

**Proof.** Let $x\in\lim_\Lambda a_\lambda$ with $\{a_\lambda\}_{\lambda\in\Lambda}\subseteq A$. Then, for every neighborhood $V$ of $x$, $(a_\lambda)_{\lambda\in\Lambda}$ is eventually in $V$. Therefore $A\cap V\neq\emptyset$. This proves that $x\in\bar{A}$. In particular, the argument applies to $\Lambda=\mathbb{N}$, i.e., when $(a_\lambda)_{\lambda\in\mathbb{N}}$ is an ordinary sequence.

On the other hand, let $x\in\bar{A}$ and let us prove the existence of a generalized sequence converging to $x$. As $x\in\bar{A}$, each neighborhood of $x$ contains at least one point of $A$. Thus, if $\mathcal{B}(x)\subseteq\mathcal{V}(x)$ is a basis of neighborhoods of $x$ directed by the usual *inverse inclusion* relation ($B_1\preccurlyeq B_2$ if, and only if, $B_1\supseteq B_2$), then $B\cap A\neq\emptyset$ for every $B\in\mathcal{B}(x)$. Hence, we can select a point $a_B\in A\cap B$ to build the generalized sequence $(a_B)_{B\in\mathcal{B}(x)}$ that converges to $x$, i.e., $x\in\lim_{\mathcal{B}(x)}a_B$ (cf. **Example 2.20**).

Moreover, if $X$ has a countable basis of neighborhoods, then there exists a *countable* filter basis $\mathcal{B}_{\mathbb{N}}(x)=\{B_n\}_{n\in\mathbb{N}}\subseteq\mathcal{V}(x)$ and the same construction produces an ordinary sequence $(a_{B_n})_{B_n\in\mathcal{B}(x)}$ converging to $x\in\bar{A}$.

### 2.1.8. Continuous functions

Let $(X,\mathcal{V}_X)$ and $(Y,\mathcal{V}_Y)$ be two topological spaces, $f$ a function from $X$ to $Y$, $x_0$ a point of $X$ and $y_0=f(x_0)\in Y$ its corresponding value under $f$.

**2.30. Definition.** We say that the function $f$ is **continuous** at $x_0$ when, for every neighborhood $V$ of $y_0:=f(x_0)$ there exists a neighborhood $U$ of $x_0$ such that $f(U)\subseteq V$. In symbols:

$$\forall V\in\mathcal{V}(y_0),\ \exists U\in\mathcal{V}(x_0)\ ::\ f(U)\subseteq V. \tag{2.5}$$

Equivalently, $f$ is continuous at $x_0$ if the inverse image of every neighborhood of $y_0$ is a neighborhood of $x_0$.

Clearly, one can replace the whole filter of neighborhoods $\mathcal{V}(y_0)$ by any filter basis of neighborhoods of $y_0$. More precisely:

**2.31. Proposition.** *The function $f: X \to Y$ is continuous at $x_0 \in X$ if, and only if, the inverse image of any neighborhood belonging to a filter basis of neighborhoods $\mathcal{B}(f(x_0))$ is a neighborhood of $x_0$. In symbols: $f$ is continuous at $x_0$ if, and only if, $f^{-1}(\mathcal{B}(f(x_0))) \subseteq \mathcal{V}(x_0)$.*

### 2.1.9. Characterization of continuity through generalized sequences

**2.32. Proposition.** *The function $f: X \to Y$ is continuous at $x_0$ if, and only if, for any generalized sequence $(x_\lambda)_{\lambda\in\Lambda}$ converging to $x_0$, the generalized sequence $(f(x_\lambda))_{\lambda\in\Lambda}$ converges to $y_0 = f(x_0)$.*

*If $X$ satisfies the first axiom of countability, then the role of generalized sequences can be replaced by* **ordinary sequences***.*

**2.33. Remark.** Although trivial, it is important to note that if $(x_\lambda)_{\lambda\in\Lambda}$ is a generalized sequence taking values in a generic set $X$ (i.e., if $x: \Lambda \to X$ is a function from the directed set $\Lambda$ into $X$), and $f: X \to Y$ is any function, then $(f(x_\lambda))_{\lambda\in\Lambda}$ is a generalized sequence in $Y$.

**Remark [on the uniqueness of the limit].** Note that no Hausdorff separation hypothesis has been made on the topological spaces $X$ and $Y$. Thus, **Proposition 2.32** must be read as follows: *the function $f: X \to Y$ is continuous at $x_0$ if, and only if, for any generalized sequence $(x_\lambda)_{\lambda\in\Lambda}$ there holds that if $x_0 \in \lim_\Lambda (x_\lambda)_{\lambda\in\Lambda}$ then $f(x_0) \in \lim_\Lambda (f(x_\lambda))_{\lambda\in\Lambda}$.*

PROOF. **[if part]** Let $f$ be continuous at $x_0$ and let $(x_\lambda)_{\lambda\in\Lambda}$ be a generalized sequence such that

$$x_0 \in \lim_\Lambda (x_\lambda)_{\lambda\in\Lambda}.$$

Let $y_0 = f(x_0)$ and consider a neighborhood $V \in \mathcal{V}(y_0)$. The continuity of $f$ shows that there exists a $U \in \mathcal{V}(x_0)$ such that $f(x) \in V$ whenever $x \in U$. Since $(x_\lambda)_{\lambda\in\Lambda} \to x_0$, we can find, in correspondence to this $U$, an index $\lambda_U \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \succcurlyeq \lambda_U$. Therefore one has $f(x_\lambda) \in V$ for every $\lambda \succcurlyeq \lambda_U$, that is, $(f(x_\lambda))_{\lambda\in\Lambda} \to f(x_0)$.

**[only if part]** Let us suppose that $f$ is not continuous at $x_0$. We show that, under this condition, there exists a generalized sequence $(x_\lambda)_{\lambda\in\Lambda}$ such that

$$(x_\lambda)_{\lambda\in\Lambda} \to x_0 \quad \textbf{but} \quad (f(x_\lambda))_{\lambda\in\Lambda} \textbf{ does not converge to } y_0 = f(x_0).$$

Indeed, the non-continuity of $f$ reads as

$$\exists V \in \mathcal{V}(y_0) :: \forall U \in \mathcal{B}(x_0) \quad f(U) \cap V^{\mathsf{C}} \neq \emptyset,$$

where $\mathcal{B}(x_0)$ is a *basis* of neighborhoods of $x_0$. Therefore, for each $U \in \mathcal{B}(x_0)$ we can pick a point $x_U$ such that $x_U \in U$ and $f(x_U) \notin V$. Clearly, the generalized sequence $(x_U)_{U\in B(x_0)}$ clearly converges to $x_0$. On the other hand, for every $U \in \mathcal{B}(x_0)$ we have $f(x_U) \in X \setminus V$ (with $V \in \mathcal{V}(y_0)$); thus, the generalized sequence $(f(x_U))_{U\in B(x_0)}$ cannot converge to $y_0 = f(x_0)$.

We use a basis of neighborhoods because, in this way, we are sure that thegeneralized sequence we build converges.

**[If $X$ has a countable local basis]** If $X$ is first countable, one can consider a countable basis $\mathcal{B}(x_0)$ to conclude.

### 2.1.10. Adherence and continuity

The following result will be often used in the sequel.

**2.34. Proposition.** *Let $f\colon X \to Y$ be a map between the topological spaces $(X, \mathcal{V}_X)$ and $(Y, \mathcal{V}_Y)$, and suppose that $f$ is* **continuous at** *$x_0 \in X$. If $x_0$ adheres to $A \subseteq X$ then $f(x_0)$ adheres to $f(A)$. In particular, if $f$ is* **continuous at every point of** *$X$ then:*

$$f(\bar{A}) \subseteq \overline{f(A)}.$$

**2.35. Remark.** Pay attention to the closure operators that appear on the left- and right-hand sides of the relation $f(\bar{A}) \subseteq \overline{f(A)}$. Although we used the *overbar* symbol to denote them, they have very different meanings. If $\mathcal{V}_1, \mathcal{V}_2$ are two topologies on the same set $X$ and $f\colon (X, \mathcal{V}_1) \to (X, \mathcal{V}_2)$ is a continuous map, then the inclusion relation reads as

$$f(\mathrm{cl}_{(X,\mathcal{V}_1)}[A]) \subseteq \mathrm{cl}_{(X,\mathcal{V}_2)}[f(A)].$$

**Proof.** Since $x_0 \in \bar{A}$, there exists a generalized sequence $(a_\lambda)_{\lambda\in\Lambda}$ of elements of $A$ which converges to $x_0$. The continúity of $f$ at $x_0$ shows that the generalized sequence $(f(a_\lambda))_{\lambda\in\Lambda}$ converges to $f(x_0)$ and this shows that $f(x_0) \in \overline{f(A)}$ because $f(x_0)$ is among the limits of a generalized sequence in $f(A)$. ■

○

**2.38. Corollary.** *Let $f\colon (X, \mathcal{V}_X) \to (Y, \mathcal{V}_Y)$ be a continuous map. If $f$ is surjective, then it sends dense subsets into dense subsets. In other words, if $\bar{A} = X$ then $\overline{f(A)} = Y$.*

**Proof.** It is sufficient to note that

$$Y = f(X) = f(\mathrm{cl}_{(X,\mathcal{V}_1)}[A]) \subseteq \mathrm{cl}_{(X,\mathcal{V}_2)}[f(A)] \subseteq Y.$$

This concludes the proof. ■

### 2.1.11. The principle of extension of the identities

**2.39. Proposition.** *Let $f, g$ be two* **continuous** *functions defined in the same topological space $X$ and taking values in the same topological space $Y$. If $Y$ is (Hausdorff) separated, then the* **coincidence set**

$$A = \{x \in X :: f(x) = g(x)\}$$

*is closed. In particular, if $X_\bullet$ is a dense subset of $X$ included in $A$, then $f \equiv g$ in the whole space $X$ because of $X = \overline{X_\bullet} \subseteq \bar{A} = A \subseteq X$.*

**Proof.** We have to prove that $\bar{A} \subseteq A$, i.e., that if $b \in \bar{A}$ then $f(b) = g(b)$. Since $b \in \bar{A}$, according to **Proposition 2.28**, there exists a generalized sequence *in* $A$, let us call it $(a_\lambda)_{\lambda\in\Lambda}$, such that $(a_\lambda)_{\lambda\in\Lambda} \to b$. Since $(f(a_\lambda))_{\lambda\in\Lambda} = (g(a_\lambda))_{\lambda\in\Lambda}$, we infer, from **Proposition 2.32**, that

$$f(b) \in \Big(\lim_{\Lambda} (f(a_\lambda))_{\lambda\in\Lambda}\Big) = \Big(\lim_{\Lambda} (g(a_\lambda))_{\lambda\in\Lambda}\Big) \ni g(b).$$

Hence $\{f(b), g(b)\} \subseteq \lim_\Lambda (f(a_\lambda))_{\lambda\in\Lambda} = \lim_\Lambda (g(a_\lambda))_{\lambda\in\Lambda}$. But since $Y$ is a Hausdorff space, the non-empty set $\lim_\Lambda (f(a_\lambda))_{\lambda\in\Lambda} = \lim_\Lambda (g(a_\lambda))_{\lambda\in\Lambda}$ is a singleton. This implies that $f(b) = g(b)$. ■

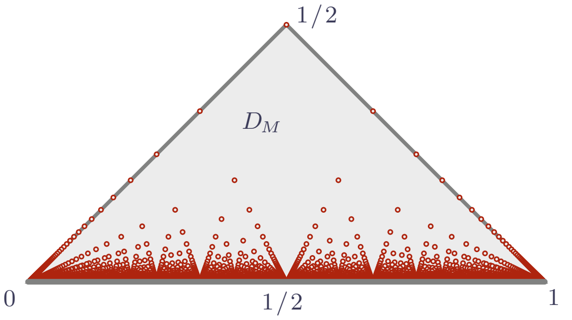



**Figure 2.3.** The Thomae function $D_M$ is periodic of period $1$ and is identically zero on the irrationals. For each rational $x := (p/q) \in \mathbb{Q}$, $(p,q) \in \mathbb{Z} \times \mathbb{N}$, the value of $D_M(p/q)$ is computed by first dividing $p$ and $q$ by $\gcd(p,q)$ to obtain the representation of $x$ in lowest terms, $x := \frac{a}{b}$, and then returning the inverse of the denominator of $\frac{a}{b}$. Note that $D_M(x) = 1$ for every $x \in \mathbb{Z}$, although this is not reported in the picture because it depicts the restriction of $D_M$ to the open interval $(0,1)$.

**2.40. Remark.** It is important to stress the range of application of principle of extension of the identities. The functions $f$ and $g$ must both be *continuous* in order to infer that if they agree on a dense subset than they agree everywhere. An example will help illustrating what we mean. Consider the case in which $X = \mathbb{R}$ is the real line. Let $f: x \in \mathbb{R} \mapsto 0 \in \mathbb{R}$ be the *continuous* function identically equal to zero in $\mathbb{R}$. Let $g: \mathbb{R} \to \mathbb{R}$ be a function (not necessarily continuous) whose restriction to a *dense* subset $A \subseteq \mathbb{R}$, $A \neq \mathbb{R}$, coincides with the function identically equal to zero:

$$g_{|A} = f_{|A} \equiv 0_{|A}.$$

If $g$ is *not* continuous in $\mathbb{R}$, then we cannot infer that $f \equiv g$ in $\mathbb{R}$, i.e., that $g \equiv 0$ in $\mathbb{R}$. This is trivial because one can always redefine $g$ outside of the dense subset $A$ in an arbitrary way to get an extension of $g_{|A}$ different from $f$. What is less trivial is that even if we discover that $g$ is a function *continuous at every point of a dense subset of* $\mathbb{R}$ then, still, *it is not necessarily* the case that $g$ is continuous in the whole of $\mathbb{R}$ and, therefore, *it is not necessarily* the case that $g \equiv f$ in $\mathbb{R}$. A concrete example of such a situation is given by the so-called **Thomae function** (sometimes also called modified Dirichlet function or the small Riemann function) defined by

$$D_M(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ \dfrac{1}{b} & \text{if } x \in \mathbb{Q} \text{ and } x = \dfrac{a}{b} \text{ if } a \in \mathbb{Z}, b \in \mathbb{N} \text{ are coprime.} \end{cases}$$

Note that $D_M(x) = 1$ for every $x \in \mathbb{Z}$. Also, note that $D_M(0) = 1$ because $b = 1$ is the only element in $\mathbb{N}$ that is coprime to $a = 0$. For each $x := \frac{p}{q} \in \mathbb{Q}$, $(p,q) \in \mathbb{Z} \times \mathbb{Z}_+$, the value of $D_M(x)$ is computed by first dividing $p$ and $q$ by $\gcd(p,q)$ to obtain the representation of $x$ in lowest terms, $x := \frac{a}{b}$, and then returning the inverse of the denominator of $\frac{a}{b}$. The function is periodic of period $1$ and bounded, $0 \leqslant D_M(x) \leqslant 1$. A sketch of the graph of $D_M$ is given in **Figure 2.3**. Clearly $D_M$ is a discontinuous function in $\mathbb{R}$. However, more precisely, it is possible to show that $D_M$ is discontinuous at the rationals *but continuous* at the irrationals. Therefore, although the function $D_M: \mathbb{R} \to \mathbb{R}$ is continuous at every point of the dense subset $\mathbb{R} \setminus \mathbb{Q}$ of $\mathbb{R}$, and $D_{M|(\mathbb{R} \setminus \mathbb{Q})} \equiv 0$, it is not the case that $g$ is continuous in the whole of $\mathbb{R}$ and, therefore, *it is not* the case that $g \equiv f$ in $\mathbb{R}$.

Note that, instead, there is no function $f$: $\mathbb{R} \to \mathbb{R}$ which is continuous on the rationals and discontinuous on the irrationals. This is because of Baire category theorem which implies that the set of continuity points of $f: \mathbb{R} \to \mathbb{R}$ must necessarily be a $G_\delta$ set (i.e., countable intersection of open sets) and it is possible to show that $\mathbb{Q}$ is not a $G_\delta$-set. On the other hand, it is simple to show that $\mathbb{R} \setminus \mathbb{Q}$ is a $G_\delta$ set. Indeed, we have $\mathbb{R} \setminus \mathbb{Q} = \cap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{q\})$.

# 3

## Topological vector spaces

**3.1. Definition.** A **topological vector space** is a vector space $\boldsymbol{\mathfrak{X}}$ endowed with a topology $\mathcal{V}_{\boldsymbol{\mathfrak{X}}}$ **compatible** with the vector space structure, that is, such that

**TVS$_1$.** The map $(x,y) \mapsto x+y$ is continuous from $\boldsymbol{\mathfrak{X}} \times \boldsymbol{\mathfrak{X}}$ to $\boldsymbol{\mathfrak{X}}$. Here, the space $\boldsymbol{\mathfrak{X}} \times \boldsymbol{\mathfrak{X}}$ is endowed with the product topology.

**TVS$_2$.** The map $(\lambda, x) \mapsto \lambda x$ is continuous from $\mathbb{K} \times \boldsymbol{\mathfrak{X}}$ into $\boldsymbol{\mathfrak{X}}$. Here, the field $\mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) is endowed with its natural euclidean topology, while the space $\mathbb{K} \times \boldsymbol{\mathfrak{X}}$ is endowed with the product topology.

Let us recast the continúity of the **scalar multiplication** and of the **vector addition** operations in terms of the filter of neighborhoods. The continúity of vector addition means that for any $(x,y) \in \boldsymbol{\mathfrak{X}} \times \boldsymbol{\mathfrak{X}}$, the following holds:

$$\forall V_{x+y} \in \mathcal{V}_{\boldsymbol{\mathfrak{X}}}(x+y) \ \exists (V_x, V_y) \in \mathcal{V}_{\boldsymbol{\mathfrak{X}}}(x) \times \mathcal{V}_{\boldsymbol{\mathfrak{X}}}(y) \quad :: \quad V_x + V_y \subseteq V_{x+y}. \tag{3.1}$$

Similarly, the continúity of scalar multiplication means that for any $(\lambda, x) \in \mathbb{K} \times \boldsymbol{\mathfrak{X}}$ there holds:

$$\forall V_{\lambda x} \in \mathcal{V}_{\boldsymbol{\mathfrak{X}}}(\lambda x) \ \exists (I_\lambda, V_x) \in \mathcal{V}_{\mathbb{K}}(\lambda) \times \mathcal{V}_{\boldsymbol{\mathfrak{X}}}(x) \quad :: \quad I_\lambda \cdot V_x \subseteq V_{\lambda x}. \tag{3.2}$$

**3.2. Remark.** Relations (3.1) and (3.2) should be keep in mind in the following operational way. If $z \in \boldsymbol{\mathfrak{X}}$, there are infinitely many ways to decompose $z$ in the form $z = x + y$ with $x, y \in \boldsymbol{\mathfrak{X}}$. The continuity of vector addition stated in (3.1) guarantess that if for my purposes it is favourable to decompose $z$ in the "simpler" form $z = x + y$, then given any $V_z \in \mathcal{V}_{\boldsymbol{\mathfrak{X}}}(z)$ there exist neighborhoods $(V_x, V_y) \in \mathcal{V}_{\boldsymbol{\mathfrak{X}}}(x) \times \mathcal{V}_{\boldsymbol{\mathfrak{X}}}(y)$ such that $V_x + V_y \subseteq V_z$. A similar observation applies to the continuity of scalar mltiplication in (3.2).

**Example 3.3.** It is simple to show that the indiscrete topology on a vector space $X$ over $\mathbb{K}$, is compatbile with the vector space structure and, threfore, a vector topology on $X$. On the other hand, a vector space $X$ over $\mathbb{K}$ endowed with the discrete topology is not a topological vector space unless $X = \{0\}$. Note that when $X = \{0\}$, the discrete topology coincides with the indiscrete topology.

**3.4. Proposition.** *The topology of a topological vector space is* **translation-invariant**. *In other terms, for every $x \in \boldsymbol{\mathfrak{X}}$ the filter of neighborhoods $\mathcal{V}(x)$ of $x$ is given by $x + \mathcal{V}(0)$. In symbols*

$$\mathcal{V}(x) = \{W \in \wp(\boldsymbol{\mathfrak{X}}) :: W = x + V \ \textbf{with} \ V \in \mathcal{V}(0)\}.$$

The proof of **Proposition 3.4** is an immediate consequence of the following simple observation:

**3.5. Lemma.** *Let $Z := X \times Y$ be the (topological) product space of $X$ and $Y$. Let $f : Z \to H$, with $H$ another topological space, be a continuous function. Then, for every $(x_0, y_0) \in Z$ the* **partial functions** *$f_{[x_0]} : y \in Y \mapsto f(x_0, y) \in H$ and $f_{[y_0]} : x \in X \mapsto f(x, y_0) \in H$ are continuous.*

**Proof.** (of **Lemma 3.5**) Let us focus on the partial function $f_{[x_0]}$. The argument to treat $f_{[y_0]}$ is the same. We define the inclusion map $\iota_{[x_0]}\colon y\in Y\to(x_0,y)\in Z$. Clearly, $f_{[x_0]}=f\circ\iota_{[x_0]}$. It is therefore sufficient to prove that $\iota_{[x_0]}$ is a continuous function. To this end we observe that for $W\subseteq Z$ we have (cf. **Figure 3.1**)

$$\begin{aligned}\iota_{[x_0]}^{-1}(W) &= \{y\in Y :: (x_0,y)\in W\}\\ &= \{y\in Y :: (x_0,y)\in W\cap(\{x_0\}\times Y)\}\\ &= \{y\in Y :: (x,y)\in W\cap(\{x_0\}\times Y) \text{ for some } x\in X\}\\ &= \pi_2(W\cap(\{x_0\}\times Y)),\end{aligned}$$

where $\pi_2\colon Z\to Y$ stands for the canonical projection on the second factor. Now, consider a generic point $(x_0,y)\in Z$ and let $(B_1\times B_2)\in\mathcal{B}(x_0)\otimes\mathcal{B}(y)$ be a basis neighborhood of $(x_0,y)$. We have $\iota_{[x_0]}^{-1}(B_1\times B_2)=\pi_2((B_1\times B_2)\cap(\{x_0\}\times Y))=\pi_2(\{x_0\}\times B_2)=B_2$. Therefore, by **Proposition 2.31** the inclusion map $\iota_{[x_0]}$ is continuous. This completes the proof.

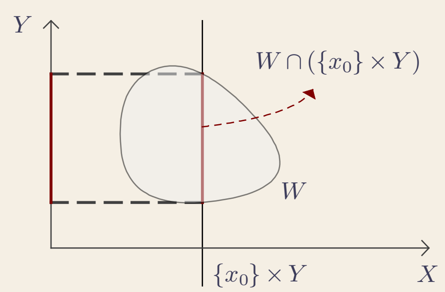



**Figure 3.1.** For $W\subseteq X\times Y$ we have $\iota_{[x_0]}^{-1}(W)=\pi_2(W\cap(\{x_0\}\times Y))$.

**Proof.** (of **Proposition 3.4**) Let us denote by $\sigma\colon(x,y)\in\mathfrak{X}\times\mathfrak{X}\mapsto x+y\in\mathfrak{X}$ the (continuous) vector sum in $\mathfrak{X}$. Let $W$ be an arbitrary neighborhood of $x\in\mathfrak{X}$. The partial function $\sigma_{[x]}$: $y\mapsto x+y$ is a bijection of $X$ **onto** $X$. According to the previous **Lemma 3.5** applied to the (continuous) vector sum $\sigma$, $\sigma_{[x]}$ is a continuous map. The inverse map $\sigma_{[x]}^{-1}\colon w\mapsto w-x$, is continuous as well, and therefore $\sigma_{[x]}$ is a (topological) homeomorphism of $X$ **onto** $X$. Since every set passing through $x$ is mapped by $\sigma_{[x]}^{-1}$ bijectively (and continuously) into a set passing through $0$, we have that $\sigma_{[x]}^{-1}$ maps bijectively every neighborhood of $x$ onto a neighborhood of $0$. In formulas, $\sigma_{[x]}^{-1}(W)\in\mathcal{V}(0)$ for every $W\in\mathcal{V}(x)$. Hence, for any $W\in\mathcal{V}(x)$ we have

$$W=\sigma_{[x]}\circ\sigma_{[x]}^{-1}(W)=x+\sigma_{[x]}^{-1}(W)\quad\text{with}\quad\sigma_{[x]}^{-1}(W)\in\mathcal{V}(0).$$

This concludes the proof.

**3.6. Remark.** Note that $\sigma_{[x]}^{-1}(W)=\{y\in X :: x+y\in W\}=-x+W$. Therefore every neighborhood $W$ of $x$ can be written as $W=x+(-x+W)$ with $-x+W\in\mathcal{V}(0)$.

By an argument similar to the one used to prove **Proposition 3.4**, one can easily show that if $x,y\in\mathfrak{X}$, $\lambda\neq0$, and $V\in\mathcal{V}(x)$, then $\lambda V\in\mathcal{V}(\lambda x)$ and $y+V\in\mathcal{V}(x+y)$.

Moreover, the following implications hold.

**3.7. Proposition.** *Let $\lambda\neq0$, $y\in\mathfrak{X}$ and $E\subseteq\mathfrak{X}$. The following assertions hold:*

*i. If $E$ is closed, then $y+E$ and $\lambda E$ are closed as well. If $E$ is compact, then $y+E$ and $\lambda E$ are compact as well.*

*ii.* *If $U$ is open then $\lambda U$ is open as well.*

*iii.* *Let $E, U \subseteq \boldsymbol{\mathfrak{X}}$. If $U$ is open than $U+E$ is open as well. In other words, given a finite number of subsets $E_1, \ldots, E_n$ of $\boldsymbol{\mathfrak{X}}$ it is suffiicent that at least one of them is open for their sum to be open. In particular, if $U$ is open, then $y+U$ is open.*

**3.8. Remark.** Note that $\{0\}$ is a compact set. However, in general, if $E$ is open (or closed) nothing can be said about $\lambda E$ when $\lambda = 0$, i.e., about the singleton $\{0\}$. It can be closed or not, but it is certainly closed when the space is Hausdorff separeted (in general, $T_1$ suffices, but this is equivalent to be $T_2$ in the category of topological vector spaces; cf. **Proposition 3.20**). In logical arguments, when one has to deal with expressions like $\Lambda E$ with $\Lambda \subseteq \mathbb{K}$, $0 \in \Lambda$, one has to treat the case $\lambda = 0$ with special care.

○

**Proof.** The assertions in *i.* and *ii.* are consequences of the homeomorphic character of scalar multiplication and of the vector addition. To prove *iii.* we observe that $U+E = \cup_{x \in E}(x+U)$. But for every $x \in E$ the set $x+U$ is open. Thus, $U+E$ is open being union of open sets. ■■■■

## 3.1 The closure of convex sets and of balanced sets

In this section we are going to prove that the (topological) closure of a balanced (resp. convex) set is still balanced (resp. convex). Before giving proofs, it is worth to observe that we are not considering how the (topological) closure of an absorbing set behaves. The reason is that it is trivially true that the closure of an absorbing set is still absorbing because any superset (in particular the closure) of an absorbing set is still absorbing (cf. **Proposition 1.26**).

**3.10. Proposition.** *In any topological vector space the following assertions hold:*

***i.*** *The closure of a balanced set is balanced;*

***ii.*** *The closure of a convex set is convex.*

***iii.*** *The closure of a vector subspace is a vector subspace.*

**Proof.** ***i.*** Let $g$ be the scalar multiplication map $(\lambda, x) \mapsto \lambda x$. By definition, a subset $A$ of a topological vector space $\boldsymbol{\mathfrak{X}}$ is balanced if $g(\mathbb{D}_\bullet \times A) \subseteq A$ (where $\mathbb{D}_\bullet$ is the closed unit disk of $\mathbb{K}$). Thus $\overline{g(\mathbb{D}_\bullet \times A)} \subseteq \bar{A}$. Also, the continúity of $g$ implies that $g(\overline{\mathbb{D}_\bullet \times A}) \subseteq \overline{g(\mathbb{D}_\bullet \times A)}$ (cf. **Proposition 2.34**). Overall,

$$g(\overline{\mathbb{D}_\bullet \times A}) \subseteq \overline{g(\mathbb{D}_\bullet \times A)} \subseteq \bar{A}.$$

On the other hand, cf. **Proposition 2.16**, $\overline{\mathbb{D}_\bullet \times A} = \mathbb{D}_\bullet \times \bar{A}$ and, therefore,

$$g(\mathbb{D}_\bullet \times \bar{A}) \subseteq \bar{A}.$$

The previous equality is nothing but the definition of «$\bar{A}$ is a balanced set».

***ii.*** For any fixed $0 \leqslant \lambda \leqslant 1$ we consider the map $f_\lambda \colon (x, y) \mapsto \lambda x + (1-\lambda) y$ defined in the topological product space $\boldsymbol{\mathfrak{X}} \times \boldsymbol{\mathfrak{X}}$ and taking values in $\boldsymbol{\mathfrak{X}}$. Note that, $f_\lambda$ is continuous because it is a composition of continuous maps. The convexity of a subset $A \subseteq \boldsymbol{\mathfrak{X}}$ can be reformulated in terms of $f_\lambda$ as

$$f_\lambda(A \times A) \subseteq A \quad \text{for every } 0 \leqslant \lambda \leqslant 1.$$

Let $0 \leqslant \lambda \leqslant 1$. The continuity of $f_\lambda$ assures (cf. **Proposition 2.34**) that $f_\lambda(\overline{A \times A}) \subseteq \bar{A}$. Since $f_\lambda(\overline{A \times A}) = f_\lambda(\bar{A} \times \bar{A})$ (cf. **Proposition 2.16**) we end up with the relation $f_\lambda(\bar{A} \times \bar{A}) \subseteq \bar{A}$. The arbitrariness of $0 \leqslant \lambda \leqslant 1$ shows that $\bar{A}$ is convex.

***ii.*** The proof is based on the same ideas already used. For any $\lambda, \mu \in \mathbb{K}$ we consider the continuous function $f_{\lambda,\mu}\colon (x, y) \in \mathfrak{X} \times \mathfrak{X} \mapsto \lambda x + \mu y \in \mathfrak{X}$. One observes that $M \subseteq \mathfrak{X}$ is a subspace of $\mathfrak{X}$ if, and only if, for any $\lambda, \mu \in \mathbb{K}$ one has $f_{\lambda,\mu}(M \times M) \subseteq M$. Hence what one has to prove is that if $f_{\lambda,\mu}(M \times M) \subseteq M$ then $f_{\lambda,\mu}(\bar{M} \times \bar{M}) \subseteq \bar{M}$. This is easy to prove. One has

$$\begin{aligned} f_{\lambda,\mu}(\bar{M} \times \bar{M}) &= f_{\lambda,\mu}(\overline{M \times M}) && (3.3)\\ &\subseteq \overline{f_{\lambda,\mu}(M \times M)} && (3.4)\\ &\subseteq \bar{M}. && (3.5) \end{aligned}$$

Indeed, (3.4) follows from the continuity of $f$, while (3.5) follows by the assumption that $f_{\lambda,\mu}(M \times M) \subseteq M$. The arbitrarity of $\lambda, \mu \in \mathbb{K}$ concludes the proof. ■

We end this section by stating some other interesting properties about the interior of a convex set.

**3.11. Proposition.** *Let $\mathfrak{X}$ be a topological vector space and $A$ a subset of $\mathfrak{X}$. If $A$ is convex then its interior $A^\circ$ is convex too.*

**Proof.** For any fixed $0 \leqslant \lambda \leqslant 1$ we consider the map $f_\lambda\colon (x, y) \mapsto \lambda x + (1 - \lambda) y$ defined in the topological product space $(\mathfrak{X} \times \mathfrak{X})$ and taking values in $\mathfrak{X}$. The convexity of a subset $A \subseteq \mathfrak{X}$ can be reformulated in terms of $f_\lambda$ as $f_\lambda(A \times A) \subseteq A$ for every $0 \leqslant \lambda \leqslant 1$. Therefore, we have to prove that $f_\lambda(A^\circ \times A^\circ) \subseteq A^\circ$. For that, we observe that since $A$ is convex (by hypothesis), we have $f_\lambda(A^\circ \times A^\circ) \subseteq f_\lambda(A \times A) \subseteq A$, so that (passing to the interiors)

$$[f_\lambda(A^\circ \times A^\circ)]^\circ \subseteq A^\circ.$$

Thus, to complete the proof, it is sufficient to prove that, for any $0 \leqslant \lambda \leqslant 1$, $f_\lambda(A^\circ \times A^\circ)$ is an open subset of $\mathfrak{X}$. But this is a particular case of **Proposition 3.7** as $f_\lambda(A^\circ \times A^\circ) = \lambda A^\circ + (1 - \lambda) A^\circ$ is the sum of two open sets. ■

**3.12. Corollary.** *Let $\mathfrak{X}$ be a topological vector space and $A$ a subset of $\mathfrak{X}$. If $A$ is open, then its convex hull is still open.*

**Proof.** Indeed, let us denote, as usual, by $K(A)$ the convex hull of $A$. According to **Proposition 3.11** we have that $[K(A)]^\circ$ is a convex set. Also, as $A$ is open, we have $A = A^\circ \subseteq K(A)$. Thus, $A = A^\circ \subseteq [K(A)]^\circ \subseteq K(A)$. Since $K(A)$ is the smallest convex set containing $A$ we conclude that $K(A) \subseteq [K(A)]^\circ$. Therefore $K(A) = [K(A)]^\circ$. ■

**3.13. Proposition.** *Let $\mathfrak{X}$ be a topological vector space and $A$ a subset of $\mathfrak{X}$. If $A$ is balanced and $0 \in A^\circ$ then its interior $A^\circ$ is balanced too.*

**Proof.** We have to show that $\mathbb{D}_\bullet A^\circ \subseteq A^\circ$. This amounts to prove that

$$\bigcup_{\lambda \in \mathbb{D}_\bullet} (\lambda A^\circ) \subseteq A^\circ.$$

Recall that (cf. **Proposition 3.7**) $\bigcup_{\lambda \in \mathbb{D}_\bullet \setminus \{0\}} (\lambda A^\circ)$ is an open set, because union of open sets. Moreover, since $A$ is balanced,

$$\bigcup_{\lambda \in \mathbb{D}_\bullet \setminus \{0\}} (\lambda A^\circ) \subseteq \mathbb{D}_\bullet A = A.$$

Passing to the interiors, we get

$$\bigcup_{\lambda\in\mathbb{D}_\bullet\setminus\{0\}}(\lambda A^\circ)\subseteq A^\circ.$$

By hypothesis, $0A^\circ=0\in A^\circ$. Therefore, $\mathbb{D}_\bullet A^\circ\subseteq A^\circ$. This concludes the proof. ■

**3.14. Remark.** Note that, if $0\notin A^\circ$ then $A^\circ$ can be not balanced. A simple example in $\mathbb{R}^2$ is depicted in **Figure 3.2**. Also, note that, in general, the interior of an absorbing set is not absorbing. For example, cf. **Figure 3.2**, the set $C^\circ:=\{x,y\in\mathbb{R}^2 :: |y|>x^2\}$ is not absorbing (it does not absorbs any point on the coordinate line $\{y=0\}$) althouh it is the interior of the absorbing set $C:=[(-1,0),(1,0)]_{\mathbb{R}}\cup\{x,y\in\mathbb{R}^2 :: |y|\geqslant x^2\}$.

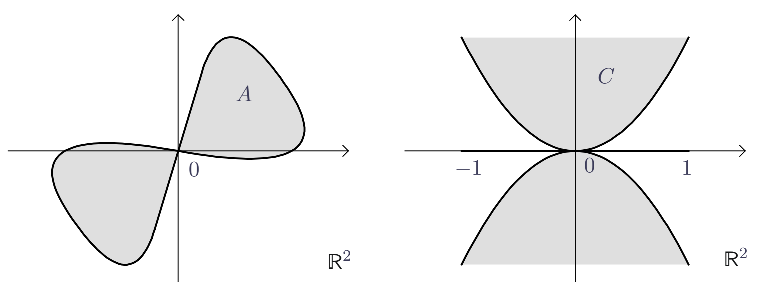



**Figure 3.2.** Left. $A\subseteq\mathbb{R}^2$ is balanced, however its interior is non-empty and does not contain the origin. Therefore, $A^\circ$ is not balanced. Right. The set $C$ is absorbing, while its interior is not.

**3.15. Proposition.** *Let $\mathfrak{X}$ be a topological vector space and $M$ a vector subspace of $\mathfrak{X}$. If $0\in M^\circ$ then the interior $M^\circ$ of $M$ is still a vector space.*

**Proof.** For any $\lambda,\mu\in\mathbb{K}$ we consider the continuous function $f_{\lambda,\mu}:(x,y)\in\mathfrak{X}\times\mathfrak{X}\mapsto\lambda x+\mu y\in\mathfrak{X}$. One observes that $M\subseteq\mathfrak{X}$ is a subspace of $\mathfrak{X}$ if, and only if, for any $\lambda,\mu\in\mathbb{K}$ one has $f_{\lambda,\mu}(M\times M)\subseteq M$. Hence what one has to prove is that if $f_{\lambda,\mu}(M\times M)\subseteq M$ then $f_{\lambda,\mu}(M^\circ\times M^\circ)\subseteq M^\circ$. To this end, we observe that

$$f_{\lambda,\mu}(M^\circ\times M^\circ)\ \subseteq\ f_{\lambda,\mu}(M\times M)\subseteq M. \tag{3.6}$$

Hence

$$[f_{\lambda,\mu}(M^\circ\times M^\circ)]^\circ\ \subseteq\ M^\circ. \tag{3.7}$$

Next, we observe that for $\lambda,\mu\in\mathbb{K}$ the set $f_{\lambda,\mu}(M^\circ\times M^\circ)=\lambda M^\circ+\mu M^\circ$ is open unless $\lambda=\mu=0$. Therefore

$$f_{\lambda,\mu}(M^\circ\times M^\circ)=[f_{\lambda,\mu}(M^\circ\times M^\circ)]^\circ\subseteq M^\circ\quad\text{for every }(\lambda,\mu)\neq(0,0). \tag{3.8}$$

It remains to check the case $(\lambda,\mu)=(0,0)$, i.e., that $f_{0,0}(M^\circ\times M^\circ)\subseteq M^\circ$. But this nothing that the assumption $0\in M^\circ$. The proof is completed. ■

**3.16. Proposition.** *Let $\mathfrak{X}$ be a topological vector space and $A$ a subset of $\mathfrak{X}$. If $A$ is absorbing and $0\in A^\circ$ then its interior $A^\circ$ is absorbing too.*

**Proof.** The proof is left as an exercise. However, compare the statement with the condition **FN**$_4$ of the structure theorem **Theorem 3.17**. ■

## 3.2 Characterization of the basis of neighborhoods of the origin of a topological vector space

We have seen that the topology of a topological vector space is invariant under translations. This means that the knowledge of a fundamental system of neighborhoods *of the origin* uniquely identifies the full topology of the space. In this section we aim to characterize the properties that a filter must possess in order to be a filter of neighborhoods of the origin for a topology *compatible with the vector space structure*. In other words, given a vector space $X$ and a filter $\mathcal{V}$ on $X$, we aim to understand which properties $\mathcal{V}$ must satisfy for the translations $(x+\mathcal{V})_{x\in X}$ to define a neighborhood topology on $X$ compatible with the vector space structure on $X$. A complete answer to this question is the content of the next result which we state without proof.

**3.17. Theorem. (Structure theorem for TVS) Assumptions:** *Let $X$ be a vector space over the field $\mathbb{K}$, $0\in X$ the origin of $X$ and $\mathcal{F}$ a filter on $X$.* **Claim:** *the family $\mathcal{F}$ is a filter of neighborhoods of the origin $0\in X$ (for a neighborhood topology $x\mapsto(x+\mathcal{F})$) compatible with the vector space structure of $X$* **if**, **and only if**, *it satisfies the following* **five properties**:

**$FN_1$. ($\mathcal{F}$ is fixed at $0$)** *The origin $0\in X$ belongs to any $V\in\mathcal{F}$. In other terms:*

$$0\in\bigcap\{V\in\mathcal{F}\}.$$

**$FN_2$. (continuity at $(0,0)$ of the vector addition)** *For every $V\in\mathcal{F}$ there exists a $W\in\mathcal{F}$ such that $W+W\subseteq V$. Compare this condition with (3.1).*

**$FN_3$. (invariance of $\mathcal{F}$ under nonzero homothétic transformations)** *For any $(\lambda,V)\in\mathbb{K}\times\mathcal{F}$ with $\lambda\neq0$, one has $\lambda V\in\mathcal{F}$.*

Homothétic transformation are also called homothéties or homogeneous dilations

**$FN_4$.** *Every $V\in\mathcal{F}$ is an* **absorbing set**.

**$FN_5$.** *Every $V\in\mathcal{F}$ contains another element of the filter $\mathcal{F}$ which is* **balanced** *(and hence also an* **absorbing set** *due to* **$FN_4$***).*

*Let us recall that a* **nonempty** *collection $\mathcal{F}$ of subsets of $X$ is* **a filter on** *$X$ if it satisfies the following three properties: The emptyset does not belong to $\mathcal{F}$; $\mathcal{F}$ is stable under finite intersections; every $V\in\wp(X)$ containing an element $U\in\mathcal{F}$ also belongs to $\mathcal{F}$. Also, recall that a filter $\mathcal{F}$ is called aa* **free filter** *if the intersection of all of its members is empty, whereas $\mathcal{F}$ is* **fixed at** *$x\in X$ when $x\in\bigcap\{V\in\mathcal{F}\}$.*

The the definition of **filter** has been given in Section 7

**3.18. Remark.** Properties **$FN_4$** and **$FN_5$**, together, assure that the family of neighborhoods $\mathcal{B}\subseteq\mathcal{F}$ consisting in of balanced (and absorbing) sets form a fundamental system of neighborhoods of the origin: $\varpi(\mathcal{B})=\mathcal{F}$.

### 3.2.1. Immediate consequences

We want to give a characterization for a topological vector space to be (Hausdorff) separated. To this end, we recall that (cf. **Axiom 2.2** and **Axiom 2.3**) a topological space $X$ is $T_1$ if whenever $x$ and $y$ are two distinct elements in $X$, there exist $V\in\mathcal{V}(x)$ and $W\in\mathcal{V}(y)$ such that $y\notin V$ and $x\notin W$. However, if $\mathfrak{X}$ is a topological vector space, then $\mathfrak{X}$ is $T_1$ if, and only if, the following (apparently weaker) property holds:

**$T_1^\circ$.** For every $z\neq0$ there exists a neighborhood $U\in\mathcal{V}(0)$ such that $z\notin U$.

Obviously $T_1$ implies $T_1^\circ$. On the other hand, assume $T_1^\circ$ holds. Given $x, y \in \mathfrak{X}$, with $x \neq y$ we set $z := x - y$. Clearly $z \neq 0$ and, therefore, by $T_1^\circ$ and **FN**$_5$ there exists a balanced set $B \in \mathcal{V}(0)$ such that $z \notin B$. We set

$$V := x + B \in \mathcal{V}(x), \quad W := y + B \in \mathcal{V}(y),$$

and show have that $x \notin W$ and $y \notin V$. Indeed, if $y = x + b$ for some $b \in B$, then $-z = y - x = b$ and, therefore, $z \in -B = B$ because $B$ is balanced. By construction, this cannot be the case (because $z \notin B$). Similarly, if $x \in W$, then $x = y + b$ for some $b \in B$. But then, $z = x - y = b$ implies that $z \in B$ and this cannot be the case because, by construction, $z \notin B$.

**3.19. Remark.** Note that $T_1^\circ$ can be equivalently stated in the following form: for every $z \neq 0$ there exists a neighborhood $U \in \mathcal{V}(0)$ such that $0 \notin z + U$. Indeed, if this is the case (i.e., if $0 \in z + U$), we can consider a balanced neighborhood of the origin $B$ included in $U$ (which necessarily satisfies $0 \notin z + B$) to infer that $z \notin B$ because, otherwise, $-z \in B$ so that $0 = z - z \in z + B$.

We can apply a similar argument to a $T_2$ topological vector space. What one finds is that the $T_2$ separation axiom is equivalent to the following (apparently weaker) assertion:

$\boldsymbol{T_2^\circ}$**.** For every $x \neq 0$ there exist neighborhoods $U \in \mathcal{V}(0)$ and $V \in \mathcal{V}(x)$ such that $U \cap V \neq \emptyset$.

We now show that in a topological vector space, the $T_1$ and $T_2$ separation axioms are equivalent.

**3.20. Proposition.** *A topological vector space $(\mathfrak{X}, \mathcal{V})$ is (Hausdorff) separated* **if, and only if**, *for every $x \neq 0$ there exists a neighborhood $U \in \mathcal{V}(0)$ of the filter of neighborhoods of the origin such that $x \notin U$. In other words, $\mathfrak{X}$ is a $T_2$ space if, and oly if, it is a $T_1$ space.*

The condition stated is referred as the $T_1$ separation property in the context of general topological spaces. In other terms, in a topological vector space $T_1 \Leftrightarrow T_2$

Before giving the proof, let us make the following observation.

**3.21. Lemma.** *Let $B$ be a balanced neighborhood of the origin and $x \in \mathfrak{X}$. If $B \cap (x + B) \neq \emptyset$ then $x \in (B - B) = (B + B)$.*

Since $B$ is balanced, we have $B = -B$. In particular, $B + B = B - B$.

PROOF. If $B \cap (x + B) \neq \emptyset$ then for some $z \in \mathfrak{X}$ we have $z \in B$ and $z \in x + B$. Hence, for some $b \in B$ we have $z = x + b$ from which $x = z - b \in B - B$. Eventually, since $B$ is balanced, we have $B = -B$. ■

**3.22. Remark.** Note that, the statement can be rephrased, by contraposition, in the following form: if $x \notin (B + B)$ then $B \cap (x + B) = \emptyset$. In the context of normed spaces this assertion can be easily understood. If $B$ is the unit ball of radius one centered at the origin, then the statement says that if $x$ is a point outside of the ball of radius 2 then the intersection of the unit ball centered at $x$ and the one centered at the origin cannot intersect.

PROOF. (of Proposition 3.20) The condition is trivially necessary. Let us show that it is sufficient too. Due to the invariance of the topology of $\mathfrak{X}$ under translations (cf. Proposition 3.4) it is sufficient to show that: **[Claim]: If** $x \neq 0$ and there exists a neighborhood $U \in \mathcal{V}(0)$ such that $x \notin U$ **then** there exists $B \in \mathcal{V}(0)$ and $V \in \mathcal{V}(x)$ such that $B \cap V = \emptyset$.

Now, from Theorem 3.17, properties **FN**$_2$ and **FN**$_5$, there exists a **balanced** set $B \in \mathcal{V}(0)$ such that $(B + B) = (B - B) \subseteq U$. From the previous Lemma 3.21 we know that if $x \notin B + B$, as in our hypothesis because of $x \notin U \supseteq B + B$, then necessarily $B \cap (x + B) = \emptyset$. It is therefore sufficient to set $V = x + B$. ■

### 3.2.2. Regularity of the topology of a topological vector space

The next result shows, in particular, that the **topology** of a topological vector space is **regular**.

**3.23. Proposition.** *In a topological vector space $(\mathfrak{X}, \mathcal{V})$ there exists a fundamental system of neighborhoods of the origin consisting of sets which are* **closed**, **balanced** *(and* **absorbing***).*

The condition of being absorbing sets is always satisfied due to property *iv.* of **Proposition** 3.17.

Proof. It is **sufficient** to prove that every neighborhood of the origin contains a **closed** and **balanced** neighborhood. Due to the properties $\mathbf{FN}_2$ and $\mathbf{FN}_5$ of the structure theorem (cf. **Theorem 3.17**), for every $U \in \mathcal{V}(0)$ there exists a **balanced** neighborhood $B$ of the origin such that

$$B + B = B - B \subseteq U.$$

Indeed $\mathbf{FN}_2$ assures the existence of a $V \in \mathcal{V}(0)$ such that $V + V \subseteq U$, while $\mathbf{FN}_5$ gives the existence of a balanced set $B \subseteq V$.

Let us show that $\bar{B} \subseteq U$ from which the result follow at once because the closure of a balanced set is still a balanced set (cf. **Proposition 3.10**). Let $x \in \bar{B}$. Every neighborhood of $x$, in particular $x + B$, intersect $B$. Since $(x + B) \cap B \neq \emptyset$, by **Lemma 3.21** we get that $x \in B + B \subseteq U$. Actually, we proved that $\bar{B} \subseteq B + B$. In fact, suppose that $x \in \bar{B}$, then $(x + B) \cap B \neq \emptyset$ and therefore $x \in B + B$ by **Lemma 3.21**. ■■■■

In the proof of **Proposition 3.23** we derived a couple of observations that deserve interest in their own right.

**3.24. Corollary.** *Let $(\mathfrak{X}, \mathcal{V})$ be a topological vector space. For every $U \in \mathcal{V}(0)$ there exists a balanced set $B \in \mathcal{V}(0)$ such that*

$$B + B = B - B \subseteq U.$$

*Also, if $B \in \mathcal{V}(0)$ is a balanced set, then*

$$\bar{B} \subseteq B + B.$$

### 3.2.3. The topology defined by a filter basis

Given a filter base $\mathcal{B}$ on a vector space $X$, we want to understand under which conditions $\mathcal{B}$ turns out to be a filter base of neighborhoods of the origin for a topology compatible with the vector space structure. The topology being, then, the one having as filter base at $x \in X$ the family $\mathcal{B}(x) := (x + B)_{B \in \mathcal{B}}$, i.e., $\mathcal{V}(x) := x + \varpi(\mathcal{B}(x))$.

The following criterion answers to this question.

**3.25. Proposition. [Assumptions]:** *Let $\mathcal{B}$ be a* **filter base** *on the vector space $X$ satisfying the following two properties:*

**$\mathbf{FB_1}$.** *Every $B \in \mathcal{B}$ is* **absorbing** *and* **balanced**.

**$\mathbf{FB_2}$.** *For every $B \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ (absorbing and balanced) such that*

$$W + W \subseteq B.$$

**[Claim]:** *Then, there* **exists** *a* **unique** *neighborhood topology on the vector space $X$ which is compatible with the vector structure of $X$, and for which $\mathcal{B}$ is a filter base of neighborhoods of the origin.*

**3.26. Remark.** The unique vector topology $\mathcal{V}: x \in X \mapsto \mathcal{V}(x) \subseteq \wp(X)$ *generated* by $\mathcal{B}$ is then defined, for any $x \in X$, by $\mathcal{V}(x) = x + \varpi(\mathcal{B})$.

PROOF. We have to show that the family $\mathcal{F} := \varpi(\mathcal{B})$ of supersets of elements in $\mathcal{B}$ satisfies the properties ***i.*** to ***v.*** of the structure theorem (cf. **Theorem 3.17**). Let us recall these properties. **FN**$_1$. ($\mathcal{F}$ is fixed at $0$) The origin $0 \in X$ belongs to any $V \in \mathcal{F}$. **FN**$_2$. (Continuity at $(0,0)$ of the vector addition) For every $V \in \mathcal{F}$ there exists a $W \in \mathcal{F}$ such that $W + W \subseteq V$. **FN**$_3$. (Invariance of $\mathcal{F}$ under nonzero homothétic transformations) For any $(\lambda, V) \in \mathbb{K} \times \mathcal{F}$ with $\lambda \neq 0$, one has $\lambda V \in \mathcal{F}$. **FN**$_4$. Every $V \in \mathcal{F}$ is an absorbing set. **FN**$_5$. Every $V \in \mathcal{F}$ contains another element of the filter $\mathcal{F}$ which is balanced (and hence also an absorbing set due to **FN**$_4$).

A closer inspection to the previous properties reveals that the only nontrivial statement to prove is **FN**$_3$ (for **FN**$_1$ recall that a balanced set can be empty, but an absorbing set must be non-empty and, therefore, must contain the origin). For that, it is sufficient to prove the following property, which is consequence of the assumptions **FB**$_1$, **FB**$_2$ and of the properties of the field $\mathbb{K}$:

**FB$_3$.** For any $(\lambda, B) \in \mathbb{K} \times \mathcal{B}$ with $\lambda \neq 0$ there exists a $B' \in \mathcal{B}$ such that $B' \subseteq \lambda B$.

Indeed, if $V$ is in $\mathcal{F}$ and $B \in \mathcal{B}$ is such that $B \subseteq V$, then $\lambda B \subseteq \lambda V$ and, by **FB$_3$**, $\lambda V$ is a superset of some $B' \in \mathcal{B}$. Hence $\lambda V \in \varpi(\mathcal{B}) = \mathcal{F}$.

Note that in **FB$_3$** we are not requiring that $\lambda B \in B$, but just that $\lambda B$ can be reached by a superset of an element in $\mathcal{B}$.

To prove the previous assertion **FB$_3$** we observe that, by **FB**$_2$, there exists a $W_1 \in \mathcal{B}$ such that $2W_1 \subseteq B$. By induction on $\mathbb{N}$, we get that for every $n \in \mathbb{N}$ there exists $W_n \in \mathcal{B}$ such that $2^n W_n \subseteq B$. Since $B$ is balanced, and for some sufficiently large $\nu \in \mathbb{N}$ one has $2^\nu > |\lambda|^{-1}$, the set $B' := W_\nu \in \mathcal{B}$ answers the question (i.e., satisfies **FB$_3$**). Indeed, by construction $|\lambda|^{-1} B' \subseteq 2^\nu B' = 2^\nu W_\nu \subseteq B$, and multiplying each side by $|\lambda|$ the assertion follows (cf. **Proposition 1.24**). ■■■■

The equality $W + W = 2W$ holds when $W$ is convex. In general one has $W + W \supseteq 2W$.

## 3.3 | Limits of generalized sequences in topological vector spaces

Let $\mathfrak{X}$ be a topological vector space, $(x_\lambda)_{\lambda \in \Lambda}$ a generalized sequence in $\mathfrak{X}$ and $x \in \lim_\Lambda x_\lambda$. As a consequence of the invariance under translations of the neighborhood topology of $\mathfrak{X}$ we get the next result.

**3.27. Proposition.** *For every generalized sequences $(x_\lambda)_{\lambda \in \Lambda}, (y_\lambda)_{\lambda \in \Lambda}$ in $\mathfrak{X}$ we have that*

$$\lim_\Lambda x_\lambda + \lim_\Lambda y_\lambda \subseteq \lim_\Lambda (x_\lambda + y_\lambda). \tag{3.9}$$

*In particular, if $\lim_\Lambda y_\lambda \neq \emptyset$ (or if $\lim_\Lambda x_\lambda \neq \emptyset$) then*

$$\lim_\Lambda (x_\lambda + y_\lambda) = \lim_\Lambda x_\lambda + \lim_\Lambda y_\lambda. \tag{3.10}$$

*Also, since enery constant generalized sequence converges, we have that for every $y \in \mathfrak{X}$ there holds $\lim_\Lambda (y + x_\lambda) = y + \lim_\Lambda x_\lambda$. Therefore*

$$x \in \lim_\Lambda x_\lambda \quad \textbf{if, and only if} \quad 0 \in \lim_\Lambda (x_\lambda - x). \tag{3.11}$$

*If $\mathfrak{X}$ is not Hausdorff separated, the previous relations have to be understood as equalities between sets, e.g.,*

$$z \in \lim_\Lambda (y + x_\lambda) \quad \Longleftrightarrow \quad z \in y + \lim_\Lambda x_\lambda. \tag{3.12}$$

**In particular**, *$\{x\} = \lim_\Lambda x_\lambda$ if, and only if, $\lim_\Lambda (x_\lambda - x) = \{0\}$. This is always the case, for example, when $\mathfrak{X}$ is Hausdorff separated.*

**PROOF. (of (3.12))** Assume $x \in \lim_\Lambda x_\lambda$ and $y \in \lim_\Lambda y_\lambda$. We then have (cf. **Proposition 2.24**)

$$(x, y) \in \lim_\Lambda (x_\lambda, y_\lambda)$$

and, therefore, by the continuity of the vector sum, we conclude that $x + y \in \lim_\Lambda (x_\lambda + y_\lambda)$. This proves (3.9). In particular, if $\lim_\Lambda x_\lambda \neq \emptyset$, then either $\emptyset = \lim_\Lambda (x_\lambda + y_\lambda)$, and in this case trivially

$$\lim_\Lambda (x_\lambda + y_\lambda) \subseteq \lim_\Lambda x_\lambda + \lim_\Lambda y_\lambda$$

from which (3.10) follows, or $\lim_\Lambda (x_\lambda + y_\lambda) \neq \emptyset$. But if $\lim_\Lambda (x_\lambda + y_\lambda) \neq \emptyset$ then, by (3.9), we have that

$$\lim_\Lambda (x_\lambda + y_\lambda) - \lim_\Lambda x_\lambda \subseteq \lim_\Lambda (x_\lambda + y_\lambda - x_\lambda) = \lim_\Lambda y_\lambda.$$

This concludes the proof because of the general remark that if $A, B, C \subseteq \mathfrak{X}$ and $A + B \subseteq C$ with $A, B \neq \emptyset$, then $A \subseteq C - B$.

This concludes the proof.

Let us show that if $A + B \subseteq C$ with $A, B \neq \emptyset$ then $A \subseteq C - B$. Indeed, let $a \in A$. By hypothesis, for every $b \in B$, there exists $c \in C$ such that $a + b = c$. Thus, $a = c - b \in C - B$. A more elegant proof is based on the observation that the Minkowski sum is associative so that $A \subseteq A + (B - B) = (A + B) - B \subseteq C - B$, where the relation $A \subseteq A + B - B$ always holds because of $0 \in B - B$. Note, however, that in general it is not true that $A \subseteq C - B$ implies $A + B \subseteq C$. Just think about balanced sets for which $B = -B$. Then it is not true that $A \subseteq C + B$ implies $A + B \subseteq C$. For example, in $\mathbb{R}^2$, if $B$ is the unit ball at the origin and $C = \{x\}$ a singleton, then it is not true, in general that $A \subseteq x + B$ implies that $A + B \subseteq \{x\}$.

**3.28. Remark.** To show that (3.12) implies (3.11), we simply observe that

$$x \in \lim_\Lambda x_\lambda \iff x \in \lim_\Lambda (x + (x_\lambda - x)) \iff x \in x + \lim_\Lambda (x_\lambda - x) \iff 0 \in \lim_\Lambda (x_\lambda - x).$$

### 3.3.1. Continúity of a bilinear map

We state the following result in the form of a Lemma since it will be needed in the sequel as a tool to prove important results in the context of locally convex spaces.

**3.29. Lemma. Assumptions:** *Let $\mathfrak{X}, \mathfrak{Y}$ and $\mathfrak{Z}$ be* **topological** *vector spaces and let $f: \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z}$ be a map* **continuous at the point** $(0, 0) \in \mathfrak{X} \times \mathfrak{Y}$. **Claim:** *If $f$ is* **bilinear** *then $f$ is continuous everywhere on $\mathfrak{X} \times \mathfrak{Y}$. In particular, let $g: \mathfrak{X} \to \mathfrak{Z}$ be continuous at $0 \in \mathfrak{X}$, if $g$ is linear, then $g$ is continuous everywhere on $\mathfrak{X}$.*

## 3.4 Complete Spaces

**3.30. Definition.** Let $\mathfrak{X}$ be a topological vector space and $\mathcal{V}(0)$ the filter of neighborhoods of the origin. Let $A$ be a subset of $\mathfrak{X}$ and $(x_\lambda)_{\lambda\in\Lambda}$ a generalized sequence in $A$. We say that the generalized sequence $(x_\lambda)_{\lambda\in\Lambda}$ is **a generalized Cauchy sequence** (or a **Cauchy net**) if for every neighborhood $U\in\mathcal{V}(0)$ there exists a $\lambda_0\in\Lambda$ such that

$$x_{\lambda_1}-x_{\lambda_2}\in U \quad \textbf{when} \quad \{\lambda_1,\lambda_2\}\succcurlyeq\lambda_0. \tag{3.13}$$

Note that, if (3.13) holds then also $x_{\lambda_2}-x_{\lambda_1}=-(x_{\lambda_1}-x_{\lambda_2})\in U$. We say that the set $A\subseteq\mathfrak{X}$ is **complete** (resp. **sequentially complete**) if every generalized Cauchy sequence (resp. every ordinary Cauchy sequence) in $A$ converges towards an element $a\in A$.

**3.31. Remark.** Note that if $(\Lambda,\preccurlyeq)$ is a **meet**-semilattice then condition (3.13) can be restated in the following form: for every neighborhood $U\in\mathcal{V}(0)$ there exists a $\lambda_*\in\Lambda$ such that $x_{\lambda_1}-x_{\lambda_2}\in U$ **whenever** $\lambda_1\wedge\lambda_2\succcurlyeq\lambda_*$.

**3.32. Remark.** If $A\subseteq\mathfrak{X}$ is a subset endowed with the subspace topology, then a generalized sequence $(a_\lambda)_{\lambda\in\Lambda}$ of elements of $A$ is a Cauchy net with respect to the subspace topology of $A$ if, and only, if $(a_\lambda)_{\lambda\in\Lambda}$ is a Cauchy net in $\mathfrak{X}$.

### 3.4.1. Simple consequences of the definition

**3.33. Proposition.** *In a topological vector space, the following assertions hold:*

***i.*** *Every convergent generalized sequence is a Cauchy net.*

***ii.*** *In a* **complete** *topological vector space any* **closed** *subset* **is complete**.

***iii.*** *In a* **Hausdorff** *topological vector space every* **complete** *subset is* **closed**.

**3.34. Remark.** Point ***ii.*** can be stated by saying that the property of being complete is (like compactness) **weakly hereditary**. Also, note that ***ii.*** claims that in a complete topological vector space the closed subsets are included in the class of complete subsets, i.e., the *closed subsets are particular complete subsets*. On the other hand ***iii.*** expresses that, when the space is Hausdorff separated, complete subsets are closed subsets. Therefore, combining ***ii.*** and ***iii.*** we get that *the class of closed subsets of a* **complete** *and Hausdorff separated topological vector space coincides with the class of its complete subsets*.

Proof. ***i.*** Let $(x_\lambda)_{\lambda\in\Lambda}$ be a convergent generalized sequence. The convergence of $(x_\lambda)_{\lambda\in\Lambda}$ means that $\lim_\Lambda x_\lambda\neq\emptyset$; let $x\in\lim_\Lambda x_\lambda$. For an arbitrary neighborhood of the origin $U\in\mathcal{V}(0)$, there exists $W\in\mathcal{V}(0)$ such that $W-W\subseteq U$. The existence of such a $W\in\mathcal{V}(0)$ comes from **Proposition 3.17** (cf. points ***ii.*** and ***v.***). Then, since $x\in\lim_\Lambda x_\lambda$, there exists $\lambda_0\in\Lambda$ (depending on $U$) such that $x_{\lambda_1}-x\in W$ and $x_{\lambda_2}-x\in W$ when $\lambda_1\succcurlyeq\lambda_0$ and $\lambda_2\succcurlyeq\lambda_0$. Hence $x_{\lambda_1}-x_{\lambda_2}=(x_{\lambda_1}-x)-(x_{\lambda_2}-x)\in W-W\subseteq U$. This shows that $(x_\lambda)_{\lambda\in\Lambda}$ is Cauchy because $U$ has been chosen arbitrarily.

Recall that if $\mathfrak{X}$ is not Hausdorff separated there can be more than one limit

Here, we are appealing to Proposition 3.27

***ii.*** Let $(a_\lambda)_{\lambda\in\Lambda}$ be a Cauchy generalized sequence in the closed subset $A\subseteq\mathfrak{X}$. We have to show that $\emptyset\neq A\text{-}\lim_\Lambda a_\lambda$. We have already seen that $A\text{-}\lim_\Lambda a_\lambda=A\cap\mathfrak{X}\text{-}\lim_\Lambda a_\lambda$ (cf. **Proposition 2.23**), in the sense that the two sets coincide when $A$ is endowed with the subspace topology. Thus, to prove that $(a_\lambda)_{\lambda\in\Lambda}$ is convergent is equivalent to prove that

$$A\cap\mathfrak{X}\text{-}\lim_{\Lambda} a_\lambda\neq\emptyset.$$

Now, $(a_\lambda)_{\lambda\in\Lambda}$ is also a Cauchy generalized sequence in $\mathfrak{X}$, and since $\mathfrak{X}$ is complete and $(a_\lambda)$ is a Cauchy net in $\mathfrak{X}$, there exists $x \in \mathfrak{X}$ such that $x \in \mathfrak{X}\text{-}\lim_\Lambda a_\lambda$. But $A$ is closed and therefore $x \in A$. Hence, $x \in A \cap \mathfrak{X}\text{-}\lim_\Lambda a_\lambda$.

***iii.*** Let $B$ be a complete subset of the **Hausdorff** topological vector space $\mathfrak{X}$. We have to prove that $\bar{B} \subseteq B$. For any $x \in \bar{B}$ there exists a generalized sequence $(b_\lambda)_{\lambda\in\Lambda}$ of elements in $B$ such that

We could use the equality symbol in $x = \lim_\Lambda b_\lambda$ because $\mathfrak{X}$ is Hausdorff separated

$$x \in \mathfrak{X}\text{-}\lim_\lambda b_\lambda.$$

Since $(b_\lambda)_{\lambda\in\Lambda}$ is convergent in $\mathfrak{X}$, according to point ***i.***, it is of the Cauchy type in $\mathfrak{X}$. But then, $(b_\lambda)_{\lambda\in\Lambda}$ is of the Cauchy type in $B$ as well. Since $B$ is complete, there exists $b \in B$ such that

$$b \in B\text{-}\lim_\lambda b_\lambda = B \cap \mathfrak{X}\text{-}\lim_\lambda b_\lambda \subseteq \mathfrak{X}\text{-}\lim_\lambda b_\lambda.$$

Hence $\{b, x\} \subseteq \mathfrak{X}\text{-}\lim_\lambda b_\lambda$ and therefore necessarily $b = x$ because $\mathfrak{X}$ is Hausdorff separated. Since $x \in \bar{B}$ is arbitrary we get $\bar{B} \subseteq B$, i.e., $B = \bar{B}$.

### 3.4.2. The principle of extension by continuity

In **Proposition 3.10** we have shown that closure of a vector subspace is a vector subspace. Combining this with **Proposition 3.33** we get the following result that we state as a Lemma just for future references.

**3.35. Lemma.** *Let $\mathfrak{X}$ be a topological vector space, $\mathfrak{M} \trianglelefteq \mathfrak{X}$ a topological vector* **subspace** *of $\mathfrak{X}$. Then the closure $\bar{\mathfrak{M}}$ is still a topological vector space. In particular, if $\mathfrak{X}$ is complete, then $\bar{\mathfrak{M}}$ is complete.*

The next result permits to extend any linear and continuous map defined on a subspace of a topological vector space, to a map defined on its closure, in such a way that it is still **linear** and **continuous**. The possibility to consider an extension which is still linear makes sense because of **Lemma 3.35**.

In a universe in which the closure of a vector space is not a vector space, would have been a non sense to talk of linear maps defined on something which we don't know to be a vector space.

**3.36. Theorem. Setting:** *Let $\mathfrak{X}$ be a topological vector space, $\mathfrak{M} \trianglelefteq \mathfrak{X}$ a topological vector* **subspace** *of $\mathfrak{X}$, and $\mathfrak{Y}$ a topological vector space.* **Assumption:** *Assume that $f: \mathfrak{M} \to \mathfrak{Y}$ is a* **linear** *and* **continuous** *map, and that the space $\mathfrak{Y}$ is* **complete and Hausdorff separated**. **Claim:** *Then, the linear map $f$ can be extended (in a unique way) to a linear and continuous map defined on $\bar{\mathfrak{M}}$.*

**3.37. Remark.** A similar result holds in the more general context of uniformly continuous maps on topological groups. For details cf. [LAWRENCE NARICI, EDWARD BECKENSTEIN, *Topological Vector Spaces*, CRC Press, **2010**| **Theorem 3.6.2**| p. 58].

**PROOF.**

Let $f, g$ be two continuous functions defined in the topological space $X$ and taking values in the topological space $Y$. If $Y$ is Hausdorff (separeted) then the set $A = \{x \in X :: f(x) = g(x)\}$ is closed. In particular, if $X_\bullet$ is a dense subset of $X$ which is included in $A$ then $f \equiv g$ in the full $X$ because $X = \overline{X_\bullet} \subseteq \bar{A} = A$.

**Uniqueness:** According to the principle of extension of the identity (cf. **Proposition 2.39**) if $g$: $\bar{\mathfrak{M}} \to \mathfrak{Y}$ is another continuous extension of $f$ then the set $\{f \equiv g\}$ is a closed subset of $\bar{\mathfrak{M}}$ which contains $\mathfrak{M}$ (because $f \equiv g$ on $\mathfrak{M}$ by assumption). The inclusions $\mathfrak{M} \subseteq \{f \equiv g\} = \overline{\{f \equiv g\}} \subseteq \bar{\mathfrak{M}}$ give $\{f \equiv g\} = \bar{\mathfrak{M}}$ because $\bar{\mathfrak{M}}$ is the smallest closed subset containing $\mathfrak{M}$.

**Existence**: The existence of such an extension of

$$f: \mathfrak{M} \to \mathfrak{Y}$$

is a consequence of the following observation.

**Claim:** *Any topological vector space induces a naturaldirected set $\Lambda$ having the following property: For every $A \subseteq \mathfrak{X}$ and any $x \in \bar{A}$, there exists a generalized sequence $(a_\lambda)_{\lambda\in\Lambda} \subseteq A$ which converges to $x$, i.e., such that $x \in \lim_\Lambda a_\lambda$.*

Note that, a crucial part of the claim is that $\Lambda$ depends on $\mathfrak{X}$ but **not** on the choices of $A \subseteq \mathfrak{X}$ and $x \in \bar{A}$.

**Proof.** It is sufficient to take $\Lambda$ as the filter $\mathcal{V}(0)$ of neighborhoods of the origin of $\mathfrak{X}$ directed by reverse inclusion. Then, given $A \subseteq \mathfrak{X}$ and $x \in \bar{A}$, we build the generalized sequence $a_\lambda : \Lambda \to A$ by choosing for every $V \in \Lambda := \mathcal{V}(0)$ an element $a_V \in (x+V) \cap A$. Clearly $x \in \lim_{\mathcal{V}} a_V$ with $(a_V)_{V\in\mathcal{V}(0)} \subseteq A$. ■

After that, let $x \in \bar{\mathfrak{M}}$. By the previous claim, there exists a generalized sequence $(m_\lambda)_{\lambda\in\Lambda} \subseteq \mathfrak{M}$, indexed by the directed set $\Lambda := \mathcal{V}(0)$, such that $x \in \lim_\Lambda m_\lambda$. The generalized sequence $(m_\lambda)_{\lambda\in\Lambda}$ is a Cauchy net in $\mathfrak{M}$. Since $f$ is linear, $(f(m_\lambda))_{\lambda\in\Lambda}$ is a Cauchy net in the complete space $\mathfrak{Y}$. Therefore $y \in \lim_\Lambda f(m_\lambda)$ for some $y \in \mathfrak{Y}$. Actually, $y = \lim_\Lambda f(m_\lambda)$ because $\mathfrak{Y}$ is Hausdorff separated.

$(m_\lambda)_{\lambda\in\Lambda}$ is convergent in $\mathfrak{X}$, therefore of Cauchy type in $\mathfrak{X}$, therefore of Cauchy type in $\mathfrak{M}$.

**Claim:** *The value $y = \lim_\Lambda f(m_\lambda)$ is well-defined as it does not depend on the generalized sequence $(m_\lambda)_{\lambda\in\Lambda}$ which converges to $x \in \bar{\mathfrak{M}}$* (as far as we consider generalized sequence all defined in the same directed set $\Lambda$). Thus, it remains well-defined the function

$$g : x \in \bar{\mathfrak{M}} \mapsto y := \lim_\Lambda f(m_\lambda) \in \mathfrak{Y}$$

**Proof.** Let $x \in \bar{\mathfrak{M}}$ and $(m_\lambda)_{\lambda\in\Lambda}, (n_\lambda)_{\lambda\in\Lambda}$ two generalized sequences such that both $(m_\lambda)_{\lambda\in\Lambda} \to x$ and $(n_\lambda)_{\lambda\in\Lambda} \to x$, i.e., such that $x \in (\lim_\Lambda m_\lambda) \cap (\lim_\Lambda n_\lambda)$. Then, we have $m_\lambda - n_\lambda \to 0$, i.e., $0 \in \lim_\Lambda (m_\lambda - n_\lambda)$. Since $f$ is continuous and linear in $\mathfrak{M}$ we get $0 = f(0) = \lim_\Lambda (f(m_\lambda - n_\lambda))_{\lambda\in\Lambda} = \lim_\Lambda f(m_\lambda) - \lim_\Lambda f(n_\lambda)$. Hence

Recall the continúity criterion in terms of generalized sequences

$$\lim_\Lambda f(m_\lambda) = \lim_\Lambda f(n_\lambda).$$

Note that the previous equalities are among elements of $\mathfrak{Y}$ because $\mathfrak{Y}$ is Hausdorff separated (by assumption). ■

**Claim:** *It is easy to show that $g$ is linear and continuous on $\bar{\mathfrak{M}}$ and that $g_{|\mathfrak{M}} \equiv f$.*

**Proof.** Let us first show that $g$ **is linear**. For any $x_1, x_2 \in \bar{\mathfrak{M}}$ there exist $(m^1_\lambda)_{\lambda\in\Lambda}, (m^2_\lambda)_{\lambda\in\Lambda}$ such that $m^1_\lambda \to x_1$ and $m^2_\lambda \to x_2$. Hence, for every $\alpha_1, \alpha_2 \in \mathbb{K}$ we have $\alpha_1 m^1_\lambda + \alpha_2 m^2_\lambda \to \alpha_1 x_1 + \alpha_2 x_2$ so that

$$\begin{aligned} g(\alpha_1 x_1 + \alpha_2 x_2) &= \lim_\Lambda f(\alpha_1 m^1_\lambda + \alpha_2 m^2_\lambda) \\ &= \alpha_1 \lim_\Lambda f(m^1_\lambda) + \alpha_2 \lim_\Lambda f(m^2_\lambda) \\ &= \alpha_1 g(x_1) + \alpha_2 g(x_2). \end{aligned}$$

Note that the argument works because we already proved that the value of $g$ does not depend on the particular generalized sequence $m : \Lambda \to \mathfrak{M}$ defined in the directed set $\Lambda$.

Let us prove the **continúity** of $g$. It is sufficient to show that $g$ is continuous at $0 \in \bar{\mathfrak{M}}$ when $\bar{\mathfrak{M}}$ is endowed with the subspace topology induced by $\mathfrak{X}$. Consider an arbitrary neighborhood of $0$ in $\mathfrak{Y}$, let us call it $W$. We have to show the existence of a neighborhood $B \in \mathcal{V}_{\mathfrak{X}}(0)$ such that

$$g(B \cap \bar{\mathfrak{M}}) \subseteq W. \tag{3.14}$$

Let $V \in \mathcal{V}_{\mathfrak{Y}}(0)$ such that

$$V + V \subseteq W.$$

Since $f$ is continuous at $0 \in \mathcal{M}$, there exists an *open* neighborhood $U^\circ$ of $0$ in $\mathcal{X}$ (i.e, $U \in \mathcal{V}_{\mathcal{X}}(0)$) such that

$$f(U^\circ \cap \mathcal{M}) \subseteq V.$$

To obtain (3.14) it is sufficient to set $B := U^\circ$, because, as we are going to prove, $g(U^\circ \cap \bar{\mathcal{M}}) \subseteq V + V$ and this will complete the proof as $V + V \subseteq W$.

Note that, the existence of such an open neighborhood is guaranteed by **Proposition 2.8**.

To this end, let $x \in U^\circ \cap \bar{\mathcal{M}}$. Since $x \in \bar{\mathcal{M}}$ there exists a generalized sequence $(m_\lambda)_{\lambda \in \Lambda}$ with values in $\mathcal{M}$ such that $m_\lambda \to x$. Since $U^\circ$ is an *open* neighborhood of $x$, one has $m_\lambda \in U$ when $\lambda \succcurlyeq \lambda_1$ for some $\lambda_1 \in \Lambda$. Hence $m_\lambda \in U^\circ \cap \mathcal{M}$ when $\lambda \succcurlyeq \lambda_1$ so that

$$f(m_\lambda) \subseteq f(U^\circ \cap \mathcal{M}) \subseteq V \quad \textbf{whenever} \quad \lambda \succcurlyeq \lambda_1.$$

On the other hand, $g(x) = \lim_\Lambda f(m_\lambda)$ so that (since $g(x) - f(m_\lambda) \to 0$, one has $g(x) - f(m_\lambda) \in V$ eventually in $\lambda$) there exists $\lambda_2 \in \Lambda$ such that

$$g(x) \in f(m_\lambda) + V \quad \textbf{whenever} \quad \lambda \succcurlyeq \lambda_2.$$

Combining the previous estimates we get that, whenever $\lambda \succcurlyeq \lambda_1 \vee \lambda_2$ we have

$$g(x) \in f(m_\lambda) + V \subseteq V + V \subseteq W.$$

Note that our directed set is a join-semilattice and therefore we can condensate the conditions $\lambda_1 \succcurlyeq \lambda$ and $\lambda_2 \succcurlyeq \lambda$ in the expression $\lambda \succcurlyeq \lambda_1 \vee \lambda_2$.

By the arbitrariness of $x \in U^\circ \cap \bar{\mathcal{M}}$, we conclude that $g(U^\circ \cap \bar{\mathcal{M}}) \subseteq W$.

It remains to prove that $g_{|\mathcal{M}} \equiv f$. But this is trivial, because if $m \in \mathcal{M}$ then the constant generalized sequence $m_\lambda := m$ converges to $m$ and, therefore, $g(m) = \lim_\Lambda f(m) = f(m)$. ■

This completes the proof. ■

### 3.4.3. Completeness in first countable topological vector spaces

Let us recall that a topological vector space is **first countable** if its topology satisfies the first axiom of countability. This amounts requiring that the topological vector space admits a **countable** filter basis of neighborhoods of the origin.

**3.38. Proposition. Assumption:** *Let* $\mathcal{X}$ *be a topological vector space* **satisfying the first axiom of countability**. **Claim:** *The space* $\mathcal{X}$ *is complete* **if**, **and only if**, *it is sequentially complete.*

**PROOF.** The nontrivial implication to prove is

$$(\textbf{sequential completeness}) \Rightarrow (\textbf{completeness}).$$

Assume $\mathcal{X}$ to be sequentially complete. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a generalized Cauchy sequence, and let $\mathcal{B} := (U_n)_{n \in \mathbb{N}}$ be a countable filter base of neighborhoods of the origin. By assumptions, for every $n \in \mathbb{N}$ there exists $\lambda_n \in \Lambda$ such that

$$x_{\mu_1} - x_{\mu_2} \in U_n \quad \textbf{whenever} \quad \{\mu_1, \mu_2\} \succcurlyeq \lambda_n.$$

We have to show that there exists $x \in \mathcal{X}$ such that $x \in \lim_\Lambda x_\lambda$. To this end, we rearrange the ordinary sequence $(\lambda_n)_{n \in \mathbb{N}}$ into a new *increasing* sequence

$$\lambda^* : n \in (\mathbb{N}, \leqslant) \mapsto \lambda_n^* \in (\Lambda, \preccurlyeq),$$

i.e., $\lambda_n^* \succcurlyeq \lambda_m^*$ if $n \geqslant m$, with the further property that

$$\lambda_n^* \succcurlyeq \lambda_i \quad \forall i \leqslant n. \tag{3.15}$$

In particular, for each $\nu \in \mathbb{N}$, we have

$$x_{\lambda_m^*} - x_{\lambda_n^*} \in U_\nu \qquad \forall m, n \geqslant \nu. \tag{3.16}$$

In fact, we have even that $x_{\lambda_m^*} - x_{\lambda_n^*} \in U_1 \cap U_2 \cap \cdots \cap U_\nu$ but we don't need this stronger remark for the proof.

The existence of a sequence $(\lambda_n^*)_{n\in\mathbb{N}}$ satisfying (3.15) can be shown by induction. Given $(\lambda_n)_{n\in\mathbb{N}}$, set $\lambda_1^* := \lambda_1$, then choose $\lambda_2^* \succcurlyeq \{\lambda_2, \lambda_1^*\}$, $\lambda_3^* \succcurlyeq \{\lambda_3, \lambda_2^*\}$, and, in general, $\lambda_n^* \succcurlyeq \{\lambda_n, \lambda_{n-1}^*\}$. Note that, by construction, if $\{\mu_1, \mu_2\} \succcurlyeq \lambda_\nu^*$ then $\{\mu_1, \mu_2\} \succcurlyeq \lambda_i$ for each $i \leqslant \nu$ and, therefore, $x_{\mu_1} - x_{\mu_2} \in U_1 \cap U_2 \cap \cdots \cap U_\nu$.

After these premises, if we define an ordinary sequence in $\mathfrak{X}$, by extracting a subsequence from $(x_\lambda)_{\lambda\in\Lambda}$ through the assignment

$$y_n := x_{\lambda_n^*} \quad \forall n \in \mathbb{N}, \tag{3.17}$$

by (3.16), we have that for each $\nu \in \mathbb{N}$

$$y_n - y_m \in U_\nu \qquad \forall m, n \geqslant \nu \tag{3.18}$$

Thus the (ordinary) sequence $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence because for every $U \in \mathcal{V}(0)$, there exists a set $U_\nu \in \mathcal{V}(0)$, $U_\nu \subseteq U$, such that $y_n - y_m \in U_\nu$ for every $m, n \geqslant \nu$.

Since $\mathfrak{X}$ is sequentially complete, $(y_n)_{n\in\mathbb{N}}$ converges to at least one point $y \in \mathfrak{X}$. Thus, the point $y \in \mathfrak{X}$ is the natural *candidate* for a possible limit of $(x_\lambda)_{\lambda\in\Lambda}$.

**Claim:** *We claim that the generalized sequence $(x_\lambda)_{\lambda\in\Lambda}$ converges to $y$. Showing this will conclude the proof of the main statement.*

Let $U \in \mathcal{V}(0)$. By the structure theorem and the first axiom of countability, there exists a balanced set $B \in \mathcal{V}(0)$ and $\nu \in \mathbb{N}$ such that

$$U_\nu + U_\nu \subseteq B + B \subseteq U. \tag{3.19}$$

We want to prove the existence of an index $\lambda_* \in \Lambda$, such that

$$x_\lambda - y \in U_\nu \quad \textbf{whenever} \quad \lambda \succcurlyeq \lambda_*. \tag{3.20}$$

To that end we use the decomposition

$$x_\lambda - y = (x_\lambda - y_n) + (y_n - y). \tag{3.21}$$

with $n$ sufficiently large so that $(y_n - y) \in U_\nu$, and it remains to prove that, eventually,

$$(x_\lambda - y_n) \in U_\nu. \tag{3.22}$$

Before proving this, let us point out that the $n$ sufficiently large has to be specified better. Indeed, in view of the next step (i.e., to prove that also $(x_\lambda - y_n) \in U_\nu$), we have to take $n > \nu$. Thus, let us agree that we choose $n > \nu$ and big enough so that $(y_n - y) \in U_\nu$.

After that, we note that $x_\lambda - y_n = x_\lambda - x_{\lambda_n^*}$ belong to $U_\nu$ as soon as $\lambda, \lambda_n^* \succcurlyeq \lambda_\nu$. But we already know that $\lambda_n^* \succcurlyeq \lambda_\nu$ because of $n > \nu$ (which implies $\lambda_n^* \succcurlyeq \lambda_\nu^* \succcurlyeq \lambda_\nu$, cf. (3.15)) Therefore, to conclude we have to set $\lambda_* := \lambda_\nu$ in (3.20).

## 3.5 The notion of bounded subset in topological vector spaces

Bounded sets, as we will see, play a central role in defining locally convex topologies on vector spaces that are in dual pair. Indeed, the polar set of a *bounded* set is a convex, balanced and absorbing set.

**3.39. Definition.** Let $\mathfrak{X}$ be a topological vector space, not necessarily Hausdorff separated. We say that a subset of $\mathfrak{X}$ is **bounded** (in $\mathfrak{X}$) if it is absorbed by every neighborhood of the origin. In other words, a set $A \subseteq \mathfrak{X}$ is bounded if, and only if,

$$\forall U \in \mathcal{V}(0),\ \exists \alpha > 0 :: A \subseteq \lambda U \textbf{ whenever } |\lambda| \geqslant \alpha. \tag{3.23}$$

**3.40. Remark.** Note that we obtain an equivalent definition if we replace the sentence «absorbed by every neighborhood of the origin» by the sentence «absorbed by every neighborhood of a fundamental system of neighborhoods of the origin» (cf. **Proposition 3.42**). In particular, according to **Proposition 3.23**, we can always test boundedness on a fundamental system of neighborhoods of the origin $\mathcal{B}(0)$ consisting of balanced sets. In this case, thanks to **Proposition 1.24**, we can replace (3.23) with the simpler condition

$$\forall B \in \mathcal{B}(0),\ \exists \alpha > 0 :: A \subseteq \alpha B.$$

**Example 3.41.** Note that, by definition, *the emptyset is always bounded*. Also note that if $\mathfrak{X}$ is bounded, then for every $U \in \mathcal{V}(0)$ there exists $\lambda \in \mathbb{K}$ such that $\mathfrak{X} \subseteq \lambda U$. Hence $\mathfrak{X} = \lambda^{-1}\mathfrak{X} \subseteq U$. But this means that $\mathfrak{X}$ is the unique neighborhood of the origin, i.e., that $\mathfrak{X}$ is endowed with the indiscrete topology (which, as it is easy to show, is compatible with every vector space structure and, therefore, a vector topology).

**Historical note.** Bounded sets of a topological vector space are also called **VON NEUMANN bounded** sets. The concept was first introduced by John von Neumann and Andrey Kolmogorov in 1935.

### 3.5.1. Immediate consequences

**3.42. Proposition.** *Let $(\mathfrak{X}, \mathcal{V})$ be a topological vector space and $A$ a subset of $\mathfrak{X}$. The set $A$ is bounded* **if**, **and only if**, *$A$ is absorbed by every neighborhood of a fundamental system of neighborhoods of the origin.*

PROOF. The **only if** part is trivial. Let us prove the **if** part. Let us denote by $\mathcal{B}(0)$ a fundamental system of neighborhoods of the origin for $\mathfrak{X}$. Let $U \in \mathcal{V}(0)$. By assumption, there exists $B \in \mathcal{B}(0)$ such that $B \subseteq U$. Since $A$ is absorbed by $B$, it is also absorbed by $U$. ■■■■

**3.43. Proposition.** *Let $\mathfrak{M}$ be a topological vector* **sub***space of $\mathfrak{X}$ and let $A \subseteq \mathfrak{M}$. The set $A$ is bounded in $\mathfrak{X}$* **if**, **and only if**, *it is bounded in $\mathfrak{M}$.*

We talk about topological vector **sub**space when the subspace is endowed with the subspace topology indced by $\mathfrak{M}$

PROOF. Suppose that $A$ is bounded in $\mathfrak{X}$ and that $A \subseteq \mathfrak{M}$. Let $U_{\mathfrak{M}} \in \mathcal{V}$ be a neighborhood of $0 \in \mathfrak{M}$ in $\mathfrak{M}$. By definition, this means that there exists some $U \in \mathcal{V}_{\mathfrak{X}}$ such that $U_{\mathfrak{M}} = \mathfrak{M} \cap U$. By assumption, there exists $\alpha > 0$ such that $A \subseteq \lambda U$ for $|\lambda| \geqslant \alpha$. But then, for every $|\lambda| \geqslant \alpha$, we also have

$$A = A \cap \mathfrak{M} \subseteq (\lambda U) \cap \mathfrak{M} = \lambda(U \cap \mathfrak{M}) = \lambda U_{\mathfrak{M}}.$$

In other words, $A$ is absorbed by $U_{\mathcal{M}}$.

On the other side, suppose that $A$ is bounded in $\mathcal{M}$. Let $U$ be a neighborhood of $0$ in $\mathcal{X}$. Clearly, $U \cap \mathcal{M}$, is a neighborhood of $0$ in $\mathcal{M}$. By hypothesis, $A$ is absorbed by $U \cap \mathcal{M}$ and therefore by $U$ too (because $U \supseteq U \cap \mathcal{M}$).

### 3.5.2. Topological operators and boundedness

**3.44. Proposition.** *In a topological vector space, the following assertions hold:*

- ***i.*** *Every* **singleton** *is a bounded subset.*
- ***ii.*** *Every* **subset** *of a bounded set is bounded. In particular, the* **intersection** *of any family of sets is bounded whenever at least one of the elements of the family is bounded.*
- ***iii.*** *The* **union of a finite number** *of bounded subsets is still a bounded subset. In particular, taking into account* ***i.****, we get that every finite subset of a topological vector space is bounded.*
- ***iv.*** *The* **closure** *of a bounded subset is still bounded.*
- ***v.*** *Every (ordinary)* **Cauchy sequence** *is bounded.*
- ***vi.*** *Every* **compact** *subset is bounded.*

**3.45. Remark.** Recall that in an (Hausdorff) separated topological space, any compact **sub**set is necessarily closed. Therefore ***vi.*** is the topological vector space counterpart of the well known result valid in metric spaces: *Let $(X,d)$ be a metric space and $A$ a subset of $X$. If $A$ is compact then $A$ is closed and bounded.*

**Proof.** ***i.*** This is a consequence of the structure theorem (cf. **Theorem 3.17**). Indeed, according to $\mathbf{FN}_4$, every neighborhood of the origin is absorbing, that is, every point of $\mathcal{X}$ is absorbed by a neighborhood of the origin.

***ii.*** This is trivial, because the property that a set absorbs another set is inherited by its supersets.

***iii.*** Again, we make use of the structure theorem (cf. **Theorem 3.17**). Indeed, let $A_1$ and $A_2$ be two bounded subsets of $\mathcal{X}$. According to $\mathbf{FN}_5$ and **Proposition 3.42**, it is sufficient to show that any *balanced* neighborhood of the origin $V \in \mathcal{V}(0)$ absorbs $A_1 \cup A_2$. To this end, it is sufficient to prove that $\alpha V \supseteq A_1 \cup A_2$ for some $\alpha > 0$ (thanks to **Proposition 1.24**, point ***ii.***). By assumption, there exist $\alpha_1 > 0, \alpha_2 > 0$ such that $\alpha_1 V \supseteq A_1$ and $\alpha_2 V \supseteq A_2$. We set $\alpha = \alpha_1 \vee \alpha_2$ and we conclude that $\alpha V \supseteq \alpha_1 V \cup \alpha_2 V \supseteq A_1 \cup A_2$.

Recall what $\mathbf{FN}_5$ states: Every $V \in \mathcal{F}$ contains another element of the filter $\mathcal{F}$ which is **balanced** (and hence also **absorbing set** due to $\mathbf{FN}_4$)

***iv.*** We make use of the regularity property of the topology as stated in **Proposition 3.23**: There exists a fundamental system of neighborhoods of the origin, let us call it $\bar{\mathcal{B}}(0)$, consisting of closed sets. Let $A$ be a bounded subset of $\mathcal{X}$ and $F$ an element of $\bar{\mathcal{B}}(0)$. By definition, since $A$ is bounded, it is absorbed by $F$. Hence, $\bar{A}$ is absorbed by $\bar{F} \equiv F$. The arbitrariness of $\bar{F}$ proves that $\bar{A}$ is bounded. The general remark here is that in a topological vector space if $B$ absorbs $A$, then $\bar{B}$ absorbs $\bar{A}$ (in fact, $A \subseteq \lambda B$ implies $A \subseteq \lambda \bar{B}$ from which $\bar{A} \subseteq \lambda \bar{B}$).

Another way is to use the relation $\overline{\lambda B} = \lambda \bar{B}$ which follows by the continuity of the scalar multiplication.

***v.*** Let $V$ be a neighborhood of the origin. There always exists (thanks to **Theorem 3.17**, conditions $\mathbf{FN}_2$ and $\mathbf{FN}_5$) a balanced neighborhood of the origin such that $B + B \subseteq V$. Since the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy, there exists $\nu \in \mathbb{N}$ such that $x_p - x_q \in B$ whenever $p, q \geqslant \nu$. As $B$ is absorbing, it absorbs $x_\nu$, that is, there exists $\alpha>0$ such that $x_\nu \in \alpha B$. Without loss of generality, **we can assume** $\alpha>1$. But then, for $p \geqslant \nu$, we have

Recall that, if $A$ is a **balanced** set then $\lambda A = |\lambda| A$ for all $\lambda \in \mathbb{K}$. Moreover, $\lambda A = |\lambda| A \subseteq |\mu| A = \mu A$ whenever $|\lambda| \leqslant |\mu|$

$$\begin{aligned} x_p \in x_\nu + B &\subseteq \alpha B + B \\ (*\text{as } B \text{ is balanced and } \alpha>1*) &\subseteq \alpha B + \alpha B \\ &= \alpha(B+B) \\ &\subseteq \alpha V. \end{aligned}$$

This shows that the set $\{x_\nu, x_{\nu+1}, \ldots\}$ is bounded. But also the set $\{x_0, x_1, \ldots, x_{\nu-1}\}$, being a finite subset, is bounded. Overall $(x_n)_{n\in\mathbb{N}}$ is bounded being the union of two bounded subsets (cf. ***iii.***).

An open and balanced neighborhood of the origini always exists. Indeed, the interior of a balanced set $B$ is balanced if $0 \in B^\circ$, and therefore the interior of a balanced neighborhood is a balanced neighborhood

***vi.*** Let $K$ be a compact subset of $\mathfrak{X}$, $U$ an open and **balanced** neighborhood of the origin. Clearly, since $U$ is also absorbing, for every $x \in \mathfrak{X}$ there exists $n \in \mathbb{N}$ such that $x \in nU$. Thus $\bigcup_{n\in\mathbb{N}} nU = \mathfrak{X}$. Hence, the family $(nU)_{n\in\mathbb{N}}$ is an open cover of the compact set $K$. We can extract from the family $(nU)_{n\in\mathbb{N}}$ a finite subcover $\{n_1U, n_2U, \ldots, n_kU\}$ of $K$. Then, we set $n_* := \sup\{n_1, \ldots, n_k\}$ and we observe that $K \subseteq n_*U$ because $U$ is balanced. Hence, $K$ is absorbed by $U$. Since the family of open and balanced neighborhoods of the origin is a fundamental system of neighborhoods of the origin, we conclude.

### 3.5.3. The continuous image of a bounded subset

**3.46. Definition.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two topological vector spaces. A map $f: \mathfrak{X} \to \mathfrak{Y}$ is called a **bounded map** if $f(B)$ is a bounded subset of $\mathfrak{Y}$ for any bounded subset $B$ of $\mathfrak{X}$. A bounded operator is not necessarily a continuous or a linear operator.

**3.47. Remark.** It is possible to show examples of nonlinear operators $f: \mathfrak{X} \to \mathbb{K}$, with $\mathfrak{X}$ a Hilbert space, which are continuous but not bounded. However, if $\mathfrak{X}$ is locally compact (and, therefore, necessarily finite-dimensional by a well-known theorem of Riesz) then the continuity of $f$ implies its boundedness. In fact, if $\mathfrak{X}$ is locally compact and $B$ is bounded, then $B$ is absorbed by a compact neighborhood of the origin $K$ whose image, by the continuity of $f$, is compact in $\mathfrak{Y}$ and, therefore, bounded in $\mathfrak{Y}$ (because of **Proposition 3.44**).

○

Given that, in general, it is not true that continuity implies boundedness, it is of some interest the next result which, in particular, applies to *seminorms* and to continuous *linear* maps.

**3.48. Proposition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two topological vector spaces, $f: \mathfrak{X} \to \mathfrak{Y}$ a map* **continuous** *continuous at $0 \in \mathfrak{X}$, and $B$ a* **bounded subset** *of $\mathfrak{X}$. The image $f(B)$ is bounded in $\mathfrak{Y}$ if at least one of the following two conditions holds:*

***i.*** *The map $f$ is* **1-homogeneous***: $f(\lambda x) = \lambda f(x)$ for every $(\lambda, x) \in \mathbb{K} \times \mathfrak{X}$.*

***ii.*** *The map $f$ is* **circled homogeneous***: $f(\lambda x) = |\lambda| f(x)$ for every $\lambda \in \mathbb{K}$.*

*Circled homogeneous maps are often referred to as* **absolutely homogeneous** *maps.*

**3.49. Remark.** In general, a seminorm on $\mathfrak{X}$ is not necessarily continous. However, the previous result implies that if a seminorm is continuous, then it maps bounded subsets of $\mathfrak{X}$ into bounded subsets of $\mathbb{R}$.

**PROOF.** Let $B$ be a bounded subset of $\mathfrak{X}$. To prove our statement, it is sufficient to show that $f(B)$ is absorbed by any balanced neighborhood of $0\in\mathfrak{Y}$. For that, since $f(0)=0$ by homogeneity, we consider a **balanced** neighborhood $V$ of $0\in\mathfrak{Y}$ and we observe that by the continuity of $f$ at $0\in\mathfrak{X}$ the preimage $U:=f^{-1}(V)$ is a neighborhood of $0$ in $\mathfrak{X}$. Since $B$ is bounded, this neighborhood $U$ has to absorb $B$. Thus, there exists $\alpha>0$ such that $B\subseteq\lambda U$ for any $|\lambda|\geqslant\alpha$ and, therefore,

$$f(B)\subseteq f(\lambda U) \quad \textbf{for any } |\lambda|\geqslant\alpha. \tag{3.24}$$

On the other hand, note that $f(\lambda U)=\lambda f(U)$ if $f$ is 1-homogeneous and $f(\lambda U)=|\lambda|f(U)$ if $f$ is absolutely homogeneous. Hence, as $f(U)\subseteq V$, in both cases we have that

$$f(\lambda U)\subseteq\lambda f(U)\cup|\lambda|f(U)\subseteq\lambda V\cup|\lambda|V. \tag{3.25}$$

But $V$ is balanced and this implies that $\lambda V=|\lambda|V$. Therefore, combining (3.24) and (3.25) we conclude that $f(B)\subseteq\lambda V$ for any $|\lambda|\geqslant\alpha$ and this proves that $f(B)$ is bounded in $\mathfrak{Y}$. ■

### 3.5.4. Mackey Lemma on bounded sets

This section is devoted to the proof Mackey[3.1] lemma concerning a remarkable property of bounded subsets in a topological vector space. The result was stated, by Mackey, as a lemma to show that a subset of a locally convex topological vector space is bounded if and only if it is weakly bounded.

**3.50. Lemma.** *Let $\mathfrak{X}$ be* **first countable** *topological vector space, i.e., a topological vector space that admits a countable filter base of neighborhoods of the origin. Let $(B_k)_{k\in\mathbb{N}}$ be a sequence of bounded subsets of $\mathfrak{X}$. Then, there exists a* **bounded** *and* **balanced** *subset $B$ of $\mathfrak{X}$ and a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive numbers, such that*

$$B_k\subseteq\lambda_k B \quad \textbf{for all } k\in\mathbb{N}.$$

**PROOF.** Let $(V_j)_{j\in\mathbb{N}}$ be a countable filter base of **balanced** neighborhoods of the origin. For every $k\in\mathbb{N}$, the set $B_k$ is bounded and therefore absorbed by any element of $(V_j)_{j\in\mathbb{N}}$. Thus, to each $k\in\mathbb{N}$, it is possible to associate a sequence $(\alpha_{k,j})_{j\in\mathbb{N}}$ of positive real numbers in such a way that $B_k\subseteq\alpha_{k,j}V_j$ for all $j\in\mathbb{N}$. Next, we set $\beta_j:=\max_{1\leqslant k\leqslant j}\alpha_{k,j}$ so that

$$B_k\subseteq\beta_j V_j \quad \textbf{for every } (k,j)\in\mathbb{N}\times\mathbb{N} :: k\leqslant j.$$

The construction is sketched in the following table.

| $\subseteq$ | $V_1$ | $V_2$ | $V_3$ | $\cdots$ | $V_n$ | $\cdots$ | $\cdots$ | $V_j$ |
|---|---|---|---|---|---|---|---|---|
| $B_1$ | $\alpha_{1,1}$ | $\alpha_{1,2}$ | $\alpha_{1,3}$ | $\cdots$ | $\alpha_{1,n}$ | $\cdots$ | $\cdots$ | $\alpha_{1,j}$ |
| $B_2$ | $\alpha_{2,1}$ | $\alpha_{2,2}$ | $\alpha_{2,3}$ | $\cdots$ | $\alpha_{2,n}$ | $\cdots$ | $\cdots$ | $\alpha_{2,j}$ |
| $B_3$ | $\alpha_{3,1}$ | $\alpha_{3,2}$ | $\alpha_{3,3}$ | $\cdots$ | $\alpha_{3,n}$ | $\cdots$ | $\cdots$ | $\alpha_{3,j}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\cdots$ | $\cdots$ | $\vdots$ |
| $B_n$ | $\alpha_{n,1}$ | $\alpha_{n,2}$ | $\alpha_{n,3}$ | $\cdots$ | $\alpha_{n,n}$ | $\cdots$ | $\cdots$ | $\alpha_{n,j}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\cdots$ | $\cdots$ | $\vdots$ |
| $B_k$ | $\alpha_{k,1}$ | $\alpha_{k,2}$ | $\alpha_{k,3}$ | $\cdots$ | $\alpha_{k,n}$ | $\cdots$ | $\cdots$ | $\alpha_{k,j}$ |

| | | | | |
|---|---|---|---|---|
| $\alpha_{1,1}$ | $\alpha_{1,2}$ | $\alpha_{1,3}$ | $\cdots$ | $\alpha_{1,n}$ |
| $\downarrow$ | $\alpha_{2,2}$ | $\alpha_{2,3}$ | $\cdots$ | $\alpha_{2,n}$ |
| $\beta_1$ | $\downarrow$ | $\alpha_{3,3}$ | $\cdots$ | $\alpha_{3,n}$ |
| | $\beta_2$ | $\downarrow$ | $\ddots$ | $\vdots$ |
| | | $\beta_3$ | | $\alpha_{n,n}$ |
| | | | | $\downarrow$ |
| | | | | $\beta_n$ |

3.1. George Whitelaw Mackey (February 1, 1916 – March 15, 2006) was an American mathematician. Mackey earned his bachelor of arts at Rice University (then the Rice Institute) in 1938 and obtained his Ph.D. at Harvard University in 1942 under the direction of Marshall H. Stone. He joined the Harvard University Mathematics Department in 1943, was appointed Landon T. Clay Professor of Mathematics and Theoretical Science in 1969 and remained there until he retired in 1985.

On the other hand, for every $k \in \mathbb{N}$ there exists $\lambda_k > 1$ such that $\alpha_{k,j} < \lambda_k \beta_j$ for every $j \leqslant k$. Hence $B_k \subseteq \alpha_{k,j} V_j \subseteq \lambda_k \beta_j V_j$ for every $(k,j) \in \mathbb{N} \times \mathbb{N} :: k \geqslant j$. This second step is sketched in the next table.

| $\subseteq$ | $V_1$ | $V_2$ | $V_3$ | $\cdots$ | $V_n$ | $\cdots$ | $\boxed{V_j}$ |
|---|---|---|---|---|---|---|---|
| $B_1$ | $\beta_1$ | $\alpha_{1,2}$ | $\alpha_{1,3}$ | $\cdots$ | $\alpha_{1,n}$ | $\cdots$ | $\alpha_{1,j}$ |
| $B_2$ | $\alpha_{2,1}$ | $\beta_2$ | $\alpha_{2,3}$ | $\cdots$ | $\alpha_{2,n}$ | $\cdots$ | $\alpha_{2,j}$ |
| $B_3$ | $\alpha_{3,1}$ | $\alpha_{3,2}$ | $\beta_3$ | $\cdots$ | $\alpha_{3,n}$ | $\cdots$ | $\alpha_{3,j}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $B_n$ | $\alpha_{n,1}$ | $\alpha_{n,2}$ | $\alpha_{n,3}$ | $\cdots$ | $\beta_n$ | $\cdots$ | $\alpha_{n,j}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $\boxed{B_k}$ | $\alpha_{k,1}$ | $\alpha_{k,2}$ | $\alpha_{k,3}$ | $\cdots$ | $\alpha_{k,n}$ | $\cdots$ | $\alpha_{k,j}$ |

| | | | | | |
|---|---|---|---|---|---|
| $\beta_1$ | $\to\lambda_1$ | | | | |
| $\alpha_{2,1}$ | $\beta_2$ | $\to\lambda_2$ | | | |
| $\alpha_{3,1}$ | $\alpha_{3,2}$ | $\beta_3$ | $\to\lambda_3$ | | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | | |
| $\alpha_{n,1}$ | $\alpha_{n,2}$ | $\alpha_{n,3}$ | $\cdots$ | $\beta_n$ | $\to\lambda_n$ |

Summarizing, we get that $B_k \subseteq \lambda_k \beta_j V_j$ **for every** $(k,j) \in \mathbb{N} \times \mathbb{N}$. Therefore, for every $k \in \mathbb{N}$

Note that, it is in order to have $B_k \subseteq \lambda_k \beta_j V_j$ **for every** $(k,j) \in \mathbb{N} \times \mathbb{N}$ that we have chosen $\lambda_k > 1$.

$$B_k \subseteq \lambda_k B \quad \textbf{with} \quad B := \cap_{j \in \mathbb{N}} \beta_j V_j.$$

The set $B$ is bounded due to **Proposition 3.42**. Indeed, $B \subseteq \beta_j V_j$ for every $j \in \mathbb{N}$, and this means that $B$ is absorbed by each of the balanced set $V_j$. This concludes the proof. ■■■■

**3.51. Remark.** The proof of Mackey lemma is straightforward in the context of normded vector space, as in any normed space there exists a countable filter base of *bounded* neighborhoods of the origin. Indeed, in the proof of Mackey lemma we have considered a filter base $(V_j)_{j \in \mathbb{N}}$ of **balanced** neighborhoods of the origin, and if at least one of the them, say $V_2$, is bounded then the proof is immediate as one has $B_k \subseteq \alpha_{k,2} V_2$ for every $k \in \mathbb{N}$ so the it is sufficient to set $B := V_2$. On the other hand, the existence of a bounded neighborhood of the origin is intimately related to the *normability* of the topological vector space. In fact, it is possible to prove that any Hausdorff separated locally convex space (see **Definition 4.1**), for which there exists a *bounded* neighborhood of the origin, is normable (cf. **Section 4.3.3** for the definition of normable topological vector space).

## 3.6 Seminorms defined on topological vector spaces

We have given the notion of seminorm in **Section 5** when we were in the (purely algebraic) context of vector spaces. Here, we remind some of the definitions in there given.

**Reminder.** Let $X$ be a vector space over the field $\mathbb{K}$ (with $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$). A function $\mathfrak{p}: X\to\mathbb{R}_0^+$ defined on the vector space $X$ is called a **seminorm** if: ▶ $\mathfrak{p}$ is **subadditive**, that is, $\mathfrak{p}(x+y)\leqslant\mathfrak{p}(x)+\mathfrak{p}(y)$ **for all** $x,y\in X$. ▶ $\mathfrak{p}$ is **circularly homogeneous**, that is, $\mathfrak{p}(\lambda x)=|\lambda|\mathfrak{p}(x)$ **for all** $\lambda\in\mathbb{K}, x\in X$. The value $\mathfrak{p}(x)$ of $\mathfrak{p}$ at $x\in X$ is often denoted by the symbol $|x|_{\mathfrak{p}}$. ▶ The sets $B_\circ:=\{x\in X :: \mathfrak{p}(x)<1\}$ and $B_\bullet:=\{x\in X :: \mathfrak{p}(x)\leqslant 1\}$ are called, respectively, the **open unit semiball** of $\mathfrak{p}$ and the **closed unit semiball** of $\mathfrak{p}$. Sometimes we shall also refer to them as the **seminormed open unit ball** and the **seminormed closed unit ball**. ▶ To stress the nomenclature is important. Indeed, the qualifications *open* and *closed* given in this context are not topological, as we are in a pure algebraic setting. Moreover, even if $X$ is endowed with a topology, it is not always the case that $\overline{B_\circ}=B_\bullet$, i.e., that the topological closure of the open unit semiball **of** $\mathfrak{p}$ coincides with the closed unit semiball **of** $\mathfrak{p}$. ▶ A seminorm $\mathfrak{p}$, such that $\mathfrak{p}(x)\neq 0$ whenever $x\neq 0$ is called a **norm on** $X$.

Note that, when $\boldsymbol{\mathfrak{X}}$ is a topological vector space and $\mathfrak{p}:\boldsymbol{\mathfrak{X}}\to\mathbb{R}_0^+$ is a seminorm, it makes sense to investigate if, and under which circumstances, $\mathfrak{p}$ is **continuous** on $\boldsymbol{\mathfrak{X}}$. Indeed, not every seminorm is necessarily continuous, as illustrated by the following example.

**Example 3.52. (Discontinuous seminorms)** We want to show that, in every infinite-dimensional normed vector space, there exist discontinuous seminorms. This is a consequence of the following result.

**3.53. Proposition.** *Let $\boldsymbol{\mathfrak{X}}$ be a nontrivial ($\boldsymbol{\mathfrak{X}}\neq\{0\}$) normed vector space over $\mathbb{K}$. The following assertions hold:* ***i.*** *A linear functional on $\boldsymbol{\mathfrak{X}}$ is discontinuous (unbounded) if, and only if, its kernel is a dense (proper) subspace.* ***ii.*** *If $(\boldsymbol{\mathfrak{X}},\|\cdot\|)$ is an* **infinite-dimensional** *normed space, then there exists a discontinuous linear function $f: X\to\mathbb{K}$.*

**3.54. Remark. Note** the stress on the «proper» subspace. This condition is necessary because otherwise the null functional would be a counterexample. Actually, condition ***i.*** can be equivalently restated as: *Any linear functional $f: X\to\mathbb{K}$, is discontinuous if, and only if, $\overline{\ker f}=\boldsymbol{\mathfrak{X}}$ and $f\not\equiv 0$.*

**Proof.** ***i.*** The result is well-known. One implication, namely that if $f$ is continuous then xor $f\equiv 0$ xor $\overline{\ker f}$ is a proper subspace of $\boldsymbol{\mathfrak{X}}$, is trivial because the kernel of a continuous functional is a closed subset. Therefore, let us prove the following statement. Namely, that *if $\overline{\ker f}$ is a proper subspace of $\boldsymbol{\mathfrak{X}}$ then $f$ is continuous and not identically zero*. To this end we use the following observation, whose proof can be found in Theorems 1.24.16/17, p. 14 in [Giles, J. R., *Introduction to the analysis of normed linear spaces*. Cambridge University Press, 2000].

**3.55. Proposition.** *In a normed vector space $(\boldsymbol{\mathfrak{X}},\|\cdot\|)$, the kernel of a nonzero linear functional is either closed or dense. Moreover, a linear functional on $\boldsymbol{\mathfrak{X}}$ is continuous if, and only if, $\ker f$ is closed.*

Indeed, once established the previous result, we can argue as follows. By the previous proposition, we know that $\ker f$ can be either closed or dense. But by assumption, $\overline{\ker f}$ is a proper subspace of $\boldsymbol{\mathfrak{X}}$ and therefore cannot be dense in $\boldsymbol{\mathfrak{X}}$. Thus, $\ker f$ must be closed. Again, by the previous proposition, $f$ is continuous.

***ii.*** Let us first recall that if $X$ is a vector space over $\mathbb{K}$, a subset of $X$ is **linearly independent** if whenever a *finite* linear combinations of elements of $B$ is zero, then all coefficients are necessarily zero. Also, we say that $B$ is a **Hamel basis** in $X$ if $B$ is linearly independent and every vector $x\in X$ can be obtained as a linear combination of vectors from $B$. This is equivalent to the condition that every $x\in X$ can be written in precisely one way as $\sum_{i\in J} c_i x_i$ where $(c_i)_{i\in I}\subseteq\mathbb{K}$, $(x_i)_{i\in I}\subseteq B$ and $J\subseteq I$ has finite cardinality.

Georg Karl Wilhelm Hamel (12 September 1877 – 4 October 1954) was a German mathematician with interests in mechanics, the foundations of mathematics and function theory. Hamel was born in Düren, Rhenish Prussia. He studied at Aachen, Berlin, Göttingen, and Karlsruhe. His doctoral adviser was David Hilbert.

Many properties of bases which hold in the finite-dimensional setting remain true in the infinite-dimensional case as well. In particular: ≣ Every vector space has a Hamel basis. In fact, every linearly independent set is contained in a Hamel basis. ≣ Any two Hamel bases of the same space have the same cardinality. ≣ Choosing images of basis vector uniquely determines a linear function, i.e., if $B$ is a basis of $X$ then for any vector space $Y$ and any map $f_B: B\to Y$ there exists exactly one linear map $f: X\to Y$ such that $f_{|B}\equiv f_B$.

After that, choose a countably infinite linearly independent set $\{x_n\}_{n\in\mathbb{N}}$ in $(\mathfrak{X}, \|\cdot\|)$ such that $\|x_n\| = 1$. A countable linearly independent set exists because $X$ is infinite-dimensional. Moreover the normalization process does not alter the linear independence of the system. Also, there exist a Hamel basis $B$ containing $\{x_n\}_{n\in\mathbb{N}}$ and a linear functional $f\colon X \to \mathbb{R}$ such that $f(x_n) = n$ for enery $n \in \mathbb{N}$ and $f(b) = 0$ for every $b \in B \setminus \{x_n\}_{n\in\mathbb{N}}$. This linear function is obviously discontinuous (unbounded) because the image under $f$ of the bounded set $\{x_n\}_{n\in\mathbb{N}}$ is not bounded.

We can now give an example of discontinuous seminorm. Consider any infinite-dimensional normed vector space $\mathfrak{X}$, and denote by $f$ a discontinuous linear functional on $\mathfrak{X}$. For every $x \in \mathfrak{X}$ we set $\mathfrak{p}(x) := |f(x)|$. Clearly $\mathfrak{p}$ is a seminorm on $\mathfrak{X}$ and $\ker \mathfrak{p} \equiv \ker f$. Also $\ker f$ is dense in $\mathfrak{X}$ because $f$ is discontinuous. It follows that $\mathfrak{p}$ cannot be continuous because, otherwise, $\ker f$ is closed other than dense in $\mathfrak{X}$ (as $\ker \mathfrak{p} \equiv \ker f$) and therefore $f \equiv 0$.

### 3.6.1. Set inclusions among unit semiballs of seminorms (continuous or not) in topological vector space

**3.56. Lemma.** *Let $\mathfrak{p}$ be a seminorm (not necessarily continuous) on a topological vector space $\mathfrak{X}$. Let $B_\circ$ and $B_\bullet$ be, respectively, the open and closed unit semiballs of $\mathfrak{p}$. We then have*

$$(B_\bullet)^\circ \subseteq B_\circ \subseteq B_\bullet \subseteq \overline{B_\circ}.$$

**Notation:** *the right superscript $^\circ$ in $(B_\bullet)^\circ$ stands for the topological interior, while the over-bar in $\overline{B_\circ}$ stands for the topological closure.*

**3.57. Remark. Roughly speaking**, if the seminorm $\mathfrak{p}$ is not continuous then the topological interior operators can remove too many points from $B_\bullet$, and the topological closure operator can add too many points to $B_\circ$.

**3.58. Remark.** From **Lemma 3.56** it also follows that $\overline{B_\circ} = \overline{B_\bullet}$ and $(B_\bullet)^\circ = (B_\circ)^\circ$. It is sufficient to pass to the closures and to the interiors in the relation $(B_\bullet)^\circ \subseteq B_\circ \subseteq B_\bullet \subseteq \overline{B_\circ}$.

**PROOF.** ▶ The inclusion $B_\circ \subseteq B_\bullet$ is trivial. It is sufficient to expand the meanings: $B_\circ = \{x \in \mathfrak{X} :: \mathfrak{p}(x) < 1\}$ and $B_\bullet = \{x \in \mathfrak{X} :: \mathfrak{p}(x) \leqslant 1\}$.

▶ Let us show that $B_\bullet \subseteq \overline{B_\circ}$. For any $x \in B_\bullet$ we have that $x/(1+\varepsilon) \in B_\circ$ for every $\varepsilon > 0$. Passing to the limit for $\varepsilon \to 0$, taking into account the continúity of the map $(\lambda, x) \mapsto \lambda x$, we get that

$$\frac{x}{1+\varepsilon} \xrightarrow{\varepsilon\to 0} x.$$

Hence, $x \in \overline{B_\circ}$.

▶ Let us show that $(B_\bullet)^\circ \subseteq B_\circ$. We want to use the same argument used for the inclusion $B_\bullet \subseteq \overline{B_\circ}$. To this end, we pass to the complement, and show that $(B_\circ)^\complement \subseteq ((B_\bullet)^\circ)^\complement$. First, note that $((B_\bullet)^\circ)^\complement = \overline{(B_\bullet)^\complement}$ so that it is sufficient to prove that

$$(B_\circ)^\complement \subseteq \overline{(B_\bullet)^\complement},$$

with $(B_\circ)^\complement = \{x \in \mathfrak{X} :: \mathfrak{p}(x) \geqslant 1\}$ and $(B_\bullet)^\complement = \{x \in \mathfrak{X} :: \mathfrak{p}(x) > 1\}$. For any $x \in (B_\circ)^\complement$ we have $(1+\varepsilon)x \in (B_\bullet)^\complement$ for every $\varepsilon > 0$. Passing to the limit for $\varepsilon \to 0$, taking into account the continúity of the map $(\lambda, x) \mapsto \lambda x$, we get that

$$(1+\varepsilon)x \xrightarrow{\varepsilon\to 0} x.$$

Hence, $x \in \overline{(B_\bullet)^\complement}$.

### 3.6.2. Properties of the unit semiballs of continuous seminorms in topological vector space

In this section we investigate the case in which $\mathfrak{X}$ is a topological vector space and $\mathfrak{p}\colon \mathfrak{X} \to \mathbb{R}_0^+$ is a **continuous** seminorm, that is, a seminorm satisfying the further condition that $\mathfrak{p}(x_\lambda) \to \mathfrak{p}(x)$ whenever $(x_\lambda)_{\lambda\in\Lambda}$ is a generalized sequence converging to $x$ (more precisely, such that $x \in \lim_\Lambda x_\lambda$).

**3.59. Proposition.** *Let $\mathfrak{p}$ be a continuous seminorm on a topological vector space. Then:*

*i.* *The open unit semiball of $\mathfrak{p}$, $B_\circ$, is a* **topologically** *open set. We already proved that it is also a convex, balanced, and absorbing set (regardless of the continúity of $\mathfrak{p}$,* cf. **Proposition 1.47**).

*ii.* *The closed unit semiball of $\mathfrak{p}$, $B_\bullet$, is a* **topologically** *closed set. We already proved that it is also a convex, balanced, and absorbing set (regardless of the continúity of $\mathfrak{p}$,* cf. **Proposition 1.47**).

*iii.* *The topological closure of $B_\circ$ is $B_\bullet$. In symbols: $\overline{B_\circ} = B_\bullet$.*

*iv.* *The topological interior of $B_\bullet$ coincides with $B_\circ$. In symbols: $(B_\bullet)^\circ = B_\circ$.*

**Notation 3.60.** *If $\mathfrak{p}$ is a continuous seminorm on the topological vector space $\mathfrak{X}$, there is no need anymore to distinguish between $B_\bullet$ and $\overline{B_\circ}$, as well as $B_\circ$ from $(B_\bullet)^\circ$. But still, one can work with two (or even a family of) continuous seminorms defined in $\mathfrak{X}$, and then it becomes important to distinguish among different unit semiballs. Therefore, we shall also denote the open unit semiball of $\mathfrak{p}$ by $B_\mathfrak{p}^\circ$ and the closed unit semiball of $\mathfrak{p}$ by $B_\mathfrak{p}^\bullet$. Sometimes, we will also use the notation $B_\bullet(\mathfrak{p})$ and $B_\circ(\mathfrak{p})$ to denote, respectively, the closed and open unit semiballs of $\mathfrak{p}$.*

**PROOF.** By assumption $\mathfrak{p}$ is a continuous function defined in $\mathfrak{X}$ and with values in the field of real numbers endowed with the standard euclidean topology.

*i.* It is sufficient to note that $B_\circ$ is the preimage (under $\mathfrak{p}$) of the open interval $(-1,1) \subset \mathbb{R}$.

*ii.* It is sufficient to note that $B_\bullet$ is the preimage (under $\mathfrak{p}$) of the closed interval $[-1,1] \subset \mathbb{R}$.

*iii.* From the inclusion $B_\circ \subseteq B_\bullet$ we get $\overline{B_\circ} \subseteq \overline{B_\bullet} = B_\bullet$ because $B_\bullet$ is a closed set. As $B_\bullet(\mathfrak{p}) \subseteq \overline{B_\circ(\mathfrak{p})}$ for every seminorm $\mathfrak{p}$ (cf. **Lemma 3.56**) we deduce that $\overline{B_\circ} = B_\bullet$.

*iv.* We have $B_\circ \subseteq B_\bullet$ and therefore $(B_\circ)^\circ = B_\circ \subseteq (B_\bullet)^\circ$, because $B_\circ$ is open. Since $(B_\bullet(\mathfrak{p}))^\circ \subseteq B_\circ(\mathfrak{p})$ for every seminorm $\mathfrak{p}$ (cf. **Lemma 3.56**) we deduce that $(B_\bullet)^\circ = B_\circ$.

### 3.6.3. Characterization of continuous seminorms

The next result, fully characterizes the continuity of a seminorm defined on a topological vector space in terms of simpler conditions.

**3.61. Proposition.** *Let $\mathfrak{p}$ be a seminorm on a topological vector space $\mathfrak{X}$. The following four assertions are equivalent:*

*i.* *The* **open** *unit semiball $B_\circ$ of $\mathfrak{p}$ is a topologically open set of $\mathfrak{X}$.*

*ii.* *The* **closed** *unit semiball $B_\bullet$ of $\mathfrak{p}$ is a* **neighborhood** *of the origin (a fortiori, due to* **Proposition 3.59**, *it must be necessarily a topologically closed neighborhood).* **In symbols:** $B_\bullet \in \mathcal{V}_\mathfrak{X}(0)$.

*iii.* *The seminorm $\mathfrak{p}$ is* **continuous at the origin**. **In terms of generalized sequences:** *If $(x_\lambda)_{\lambda\in\Lambda} \to 0$ then $(\mathfrak{p}(x_\lambda))_{\lambda\in\Lambda} \to \mathfrak{p}(0) = 0$.*

Recall that for any seminorm $\mathfrak{p}$ we necessarily have $\mathfrak{p}(0) = 0$

*iv.* *The seminorm* $\mathfrak{p}$ *is continuous everywhere in* $\mathfrak{X}$.

**PROOF.**

**[*i.* implies *ii.*]** The closed unit semiball contains the open unit semiball $B_\circ$ which is an open neighborhood of the origin. Hence, $B_\bullet$ is a neighborhood of the origin.

**[*ii.* implies *iii.*]** Let $J_\varepsilon := [-\varepsilon, \varepsilon]$ be a neighborhood of $0_\mathbb{R}$ in $\mathbb{R}$. The preimage of $J_\varepsilon$ under $\mathfrak{p}$ is the semiball $\varepsilon B_\bullet$ which is a neighborhood of $0$ in $\mathfrak{X}$ due to the invariance of $\mathcal{V}_\mathfrak{X}(0)$ under non-zero homotheties. Hence, $\mathfrak{p}$ is continuous at $0 \in \mathfrak{X}$.

**[*iii.* implies *iv.*]** Let $(x_\lambda)_{\lambda\in\Lambda}$ be a generalized sequence converging to $x \in \mathfrak{X}$. Since $(x_\lambda)_{\lambda\in\Lambda} \to x$ in $\mathfrak{X}$, $(x_\lambda - x)_{\lambda\in\Lambda} \to 0$ in $\mathfrak{X}$. But then, $\mathfrak{p}(x_\lambda - x) \to 0$ because, by assumption, $\mathfrak{p}$ is continuous at $0 \in \mathfrak{X}$. By the reverse triangular inequality, we get

$$|\mathfrak{p}(x_\lambda) - \mathfrak{p}(x)| \leqslant \mathfrak{p}(x_\lambda - x) \to 0.$$

Hence, $\mathfrak{p}(x_\lambda) \to \mathfrak{p}(x)$.

**[*iv.* implies *i.*]** It is an immediate consequence of **Proposition 3.59**.

### 3.6.4. The gauge of a barrel set (tonneau)

Let us start by giving the definition of a **barrel** set, also referred to as a **tonneau** in Bourbaki's terminology.

**3.62. Definition.** In a topological vector space, we call **barrel** (or **barrelled set**, or **tonneau**) every set which is **absorbing**, **balanced**, **convex** and **closed**.

**3.63. Remark.** Note that the definition of barrelled set cannot be given in a purely algebraic vector space, simply because we require a barrel to be (topologically) closed. All the other requirements are, instead, purely algebraic.

**Example 3.64.** The closed unit ball $B_\mathfrak{p}^\bullet$ of a continuous seminorm $\mathfrak{p}$ is a barrelled set. More generally, the topological closure of the open (or closed) unit semiball $\overline{B_\circ} = \overline{B_\bullet}$ of a (not necessarily continuous) seminorm $\mathfrak{p}$ is a barrelled set (because the topological closure retains the properties of being absorbing, balanced, and convex).

**3.65. Proposition.** *Let* $T$ *be a barrel in a topological vector space* $\mathfrak{X}$. *Then:*

*i.* *There exists a seminorm* $\mathfrak{p}$ *(not necessarily continuous) on* $\mathfrak{X}$, *and just one, such that* $B_\bullet$ *coincides with* $T$. *Such a seminorm is nothing but the gauge* $\mathfrak{p}_T$ *of* $T$. *We say that* $\mathfrak{p}$ *is the seminorm generated by the barrel* $T$.

*ii.* *The seminorm generated by the barrel* $T$ *is continuous if, and only if,* $T$ *is a neighborhood of the origin.*

**3.66. Remark.** It is important to compare **Proposition 3.65** with the content of **Proposition 1.51**. In **Proposition 1.51** we showed that if $A$ is an absorbing, balanced, and convex subset of a vector space $X$, then the gauge $\mathfrak{p}_A$ of $A$ is a seminorm. But in general, this seminormdoes not retain full information about $A$, in the sense that if we only know $\mathfrak{p}_A$ then we do not know if $\mathfrak{p}_A$ come from $A$ or any other subset $B$ in between $B_\circ(\mathfrak{p}_A)$ and $B_\bullet(\mathfrak{p}_A)$. Here, in the context of topological vector spaces, we have that if $A$ is also closed, then $A$ can be recovered through its gauge, because, in this case, $\mathfrak{p}_A = B_\bullet(\mathfrak{p}_A)$.

**Proof.** ***i.*** Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two seminorms having $T$ as closed unit semiball, that is, such that

$$B_\bullet(\mathfrak{p}_1) = B_\bullet(\mathfrak{p}_2) = T.$$

Due to **Corollary 1.53**, necessarily $\mathfrak{p}_1 \equiv \mathfrak{p}_2$. This shows the **uniqueness** of the seminorm.

**Let us show the existence.** It is sufficient to take $\mathfrak{p}$ as the gauge of $T$, that is, to set

$$\mathfrak{p} := \mathfrak{p}_T.$$

Indeed, we already showed, in **Proposition 1.51**, that $\mathfrak{p}_T$ is a seminorm on $\mathfrak{X}$ such that

$$B_\circ(\mathfrak{p}_T) \subseteq T \subseteq B_\bullet(\mathfrak{p}_T).$$

Taking the topological closure we then get $\overline{B_\circ(\mathfrak{p}_T)} \subseteq \bar{T} = T \subseteq B_\bullet(\mathfrak{p}_T)$. On the other hand, **Lemma 3.56** shows that $B_\bullet(\mathfrak{p}_T) \subseteq \overline{B_\circ(\mathfrak{p}_T)}$. Hence, $\overline{B_\circ(\mathfrak{p}_T)} \subseteq T \subseteq B_\bullet(\mathfrak{p}_T) \subseteq \overline{B_\circ(\mathfrak{p}_T)}$ and this shows that $T = B_\bullet(\mathfrak{p}_T) = \overline{B_\circ(\mathfrak{p}_T)}$.

***ii.*** It is a consequence of the characterization of continuous seminorms stated in **Proposition 3.61**. Indeed, $\mathfrak{p} \equiv \mathfrak{p}_T$ is continuous, **if, and only if**, the barrelled set $T$, which coincides with $B_\bullet(\mathfrak{p}_T)$, is a neighborhood of the origin $0 \in \mathfrak{X}$. ■■■■

Reminder (Corollary 1.53) Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two seminorms defined on the same vector space $X$ and having the **same closed unit semiball** (or the **same open unit semiball**). Then, the two seminorms are identical: $\mathfrak{p}_1 \equiv \mathfrak{p}_2$.

Reminder (Proposition 1.51) Let $X$ be a vector space and $A \subseteq X$. The following assertions hold: ***i.*** If $A$ is the **open (or closed) unit semiball** of a seminorm $\mathfrak{p}$ on $X$, then the gauge of $A$ coincides with $\mathfrak{p}$. In other terms: $\mathfrak{p} \equiv \mathfrak{p}_A$. ***ii.*** If $A$ is a **convex**, **balanced** and **absorbing** subset of $X$, then the gauge $\mathfrak{p}_A$ induced by $A$ is a **seminorm**. Moreover, $B_\circ(\mathfrak{p}_A) \subseteq A \subseteq B_\bullet(\mathfrak{p}_A)$.

### 3.6.5. Equivalence of the barrelled neighborhoods of the origin and the closed unit balls of continuous seminorms

Given a topological vector space $\mathfrak{X}$, we call **barrelled neighborhood of the origin** any barrel belonging to $\mathcal{V}_{\mathfrak{X}}(0)$.

**3.67. Proposition.** *Let $T$ be a subset of $\mathfrak{X}$. The following two statements are equivalent:*

***i.*** *The set $T$ is a barrelled neighborhood of the origin;*

***ii.*** *The set $T$ is the closed unit semiball of some* **continuous** *seminorm on $\mathfrak{X}$ (namely, of the gauge $\mathfrak{p}_T$ of $T$).*

**Proof.** It is a direct consequence of **Proposition 3.61** and **Proposition 3.65**. ■■■■

# 4

# Locally Convex (Topological Vector) Spaces

**History.** Metrizable topologies on vector spaces have been studied since their introduction in Maurice Frechet's 1902 Ph.D. thesis *Sur quelques points du calcul fonctionnel* (wherein the notion of a metric was first introduced). More precisely, on p. 18, he writes:

> «Considérons une classe $(V)$ d'éléments de nature quelconque, mais tels qu'on sache discerner si deux d'entre eux sont ou non identiques et tels, de plus, qu'fi deux quelconques d'entre eux $A,B$, on puisse faire correspondre un nombre $(A,B)=(B,A)\geqslant 0$ qui jouit des deux propriétés suivantes: 1° La condition néessaire et suffisante pour que $(A,B)$ soit nul est que $A$ et $B$ soient identiques. 2° II existe une fonction positive bien déterminé $f(\varepsilon)$ tendant vers zéro avec $\varepsilon$, telle que les inégalités $(A,B)\leqslant\varepsilon$, $(B,C)\leqslant\varepsilon$ entrainent $(A,C)\leqslant f(\varepsilon)$, quels que soient les éléments $A,B,C$. Autrement dit, il suffit que $(A,B)$ et $(B,C)$ soient petits pour qu'il en soit de méme de $(A,C)$. Nous appellerons *voisinage* de $A$ et de $B$ le nombre $(A,B)$.»

Felix Hausdorff introduced the notion of topological space in 1914. Although some mathematicians implicitly used locally convex topologies, it dates back to von Neumann, in 1935, the general definition of a locally convex space (called a convex space by him). For further details, we refer to [von Neumann, J. *Collected works*, Vol II. p. 508-52] and [Dieudonne, J. *History of Functional Analysis*, Chapter VIII, Section 2].

## 4.1 | Definition, construction, and characterization of locally convex spaces

**4.1. Definition.** Let $X$ be a vector space. We say that a topology on $X$ is **locally convex at** $x\in X$, if it admits a fundamental system (basis) of neighborhoods of $x$ consisting of convex sets. We say that a topology on $X$ is **locally convex** if it is locally convex at **any** $x\in X$. Note that we do not require a locally convex topology any compatibility with the vector space structure on $X$.

A **topological vector space** $(\mathfrak{X},\mathcal{V})$ is called a **locally convex space** (**LC TVS** space) if its neighborhood topology $\mathcal{V}$ is locally convex. It is immediate to see that, equivalently, $(\mathfrak{X},\mathcal{V})$ is a locally convex space if it admits a filter base of neighborhoods (just) **of the origin** consisting of convex sets, i.e., if $\mathcal{V}$ is locally convex at $0_{\mathfrak{X}}$.

For topological vector space to be a locally convex **space** it is sufficient to impose that its topology is locally convex at the origin.

**Example 4.2.** Every normed vector space is a locally convex space. A locally convex topology on a vector space $X$ needs not to be compatible with the vector space structure. A simple example arises when $X$ is a nontrivial vector space and $\mathcal{V}$ the discrete topology on $X$. In this case, $(X,\mathcal{V})$ is not a topological vector space. Nevertheless, for every $x\in X$, the singleton $\{\{x\}\}$ is a fundamental system of neighborhoods of $x$ consisting of convex sets.

It is natural to investigate under which conditions a filter base on a vector space, consisting of absorbing, balanced and convex sets, induces a locally convex topology compatible with the vector space structure. In that regard, we have the following result.

**4.3. Proposition. Assumptions:** *Let $X$ be a (purely algebraic) vector space and $\mathcal{B}$ a* **filter base** *on $X$ consisting of sets which are at the same time, absorbing, balanced, and convex (note that any element of $\mathcal{B}$, being absorbing, has to pass through the origin).*

**Claim:** *The filter base $\mathcal{B}$ is a fundamental system of neighborhoods of the origin for a topology on $X$ compatible with the vector structure on $X$ (and locally convex),* **if, and only if**,

$$\forall U \in \mathcal{B},\ \forall \rho \in \mathbb{R}_+^*, \quad \exists W \in \mathcal{B} :: W \subseteq \rho U. \tag{4.1}$$

*Note that, in order to satisfy (4.1), $W$ can be chosen dependent both on $\rho$ and $U$.*

**4.4. Remark.** Note that if $\mathcal{S}$ is any family of subsets of $X$ then the family

$$\mathcal{B} := \bigcup_{\lambda \in \mathbb{R}_+^*} \lambda \mathcal{S} := \{\lambda S :: (\lambda, S) \in \mathbb{R}_+^* \times \mathcal{S}\}$$

satisfies (4.1). Indeed, if $U \in \mathcal{B}$ then $U = \lambda S$ for some $(\lambda, S) \in \mathbb{R}_+^* \times \mathcal{S}$ and therefore, for any $\rho > 0$ it sufficient to set $W = \rho\lambda S$ to get $W \subseteq \rho U$ (actually, $W = \rho U$). Note that we are not assuming any geometric hypothesis on the elements of $\mathcal{B}$ (e.g., absorbing, balanced, or convex).

On the other hand, in general, the family

$$\mathcal{B}_\sharp := \bigcup_{n \in \mathbb{N}} \frac{1}{n}\mathcal{S} := \left\{\frac{1}{n}S :: (n, S) \in \mathbb{N} \times \mathcal{S}\right\}$$

doe not satisfy (4.1).

For example, if $X = \mathbb{R}^2$ and $\mathcal{S} = \{\mathbb{S}^1\}$ then given $\mathbb{S}^1 \in \mathcal{B}_\sharp$ and $\rho = \pi$ there exists no element $\frac{1}{n}\mathbb{S}^1 \in \mathcal{B}_\sharp$ such that $\frac{1}{n}\mathbb{S}^1 \subseteq \pi\mathbb{S}^1$. However, if $\mathcal{S} = \{\mathbb{D}_\bullet\}$ then given any $\frac{1}{n}\mathbb{D}_\bullet \in \mathcal{B}_\sharp$ and any $\rho > 0$, it is sufficient to set $W = \frac{1}{\lceil n/\rho \rceil}\mathbb{D}_\bullet$ to get that $W \subseteq \frac{1}{n/\rho}\mathbb{D}_\bullet = \frac{\rho}{n}\mathbb{D}_\bullet$. Note that the singleton $\{\mathbb{D}_\bullet\}$, as any other singleton consisting of a nonempty set, is a filter base (cf. **Example 1.71**).

Note that $\mathbb{D}_\bullet$ is a balanced set and, in fact, everything always works when $\mathcal{S}$ is a family of balanced subsets of $X$: if $\mathcal{S}$ is any family of *balanced* subsets of $X$, then $\mathcal{B}_\sharp$ satisfies (4.1). Indeed, given any $\frac{1}{n}S \in \mathcal{B}_\sharp$ and any $\rho > 0$, it is sufficient to set $W = \frac{1}{\lceil n/\rho \rceil}S$ to get that $W \subseteq \frac{\rho}{n}S$.

**Proof. Necessity:** If $X$ is a locally convex topological vector space then, $\rho U$ is an element of $\mathcal{V}(0)$ for any $(\rho, U) \in \mathbb{R}_+^* \times \mathcal{V}(0)$ (this is nothing but the homothétic invariance of $\mathcal{V}(0)$ stated in the structure **Theorem 3.17**). Since $\mathcal{B}$ is a filter basis for $\mathcal{V}(0)$, and $\rho U \in \mathcal{V}(0)$, $\rho U$ contains an element $W \in \mathcal{B}$.

**Sufficiency (assume that (4.1) holds):** We will make use of **Proposition 3.25** whose statement we recall here:

*If the filter base $\mathcal{B}$ is such that:* **FB$_1$.** *Every $U \in \mathcal{B}$ is absorbing and balanced;* **FB$_2$.** *For every $U \in \mathcal{B}$ there is $W \in \mathcal{B}$ (absorbing and balanced) such that $W + W \subseteq U$;* **then**, *there* **exists**, *and is* **unique**, *a topology on $X$ that is compatible with the vector structure of $X$ and for which $\mathcal{B}$ is a filter base of neighborhoods of the origin.*

Clearly, we only have to prove that if (4.1) holds, then **FB$_2$** holds. For that, let $U \in \mathcal{B}$. By hypothesis (take $\rho := 1/2$), there exists an element $W \in \mathcal{B}$ such that $W \subseteq \frac{1}{2}U$. But then,

$$W + W \subseteq \frac{1}{2}U + \frac{1}{2}U = U,$$

where the last equality is a consequence of the convexity assumption made on any $U \in \mathcal{B}$ (because, in general, one only has $U \subseteq \frac{1}{2}U + \frac{1}{2}U$). Therefore, for every $U \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ such that $W + W \subseteq U$. ■■■■

From the previous characterization, we get the following criterion, useful to generate locally convex spaces.

**4.5. Proposition. Assumptions:** *Let $X$ be a (purely algebraic) vector space and $\mathcal{S}$ a* **filter base** *on $X$ consisting of sets that are at the same time absorbing, balanced, and convex.*

**Claim:** *The family $\mathcal{B} := \bigcup_{\lambda \in \mathbb{R}_+^*} \lambda \mathcal{S}$, consisting of the sets obtained by the elements of $\mathcal{S}$ via any homothétic transformation of strictly positive ratio, is still a filter base and, in fact, a fundamental system of neighborhoods of the origin for a locally convex topology on $X$ compatible with the vector space structure of $X$.*

**4.6. Definition.** The couple $(X, \mathcal{V})$ with $\mathcal{V}(0) := \varpi(\mathcal{B})$ the vector topology generated by the fundamental system of neighborhoods of the origin $\mathcal{B} := \bigcup_{\lambda \in \mathbb{R}_+^*} \lambda \mathcal{S}$ as in **Proposition 4.5**, is called the locally convex space generated by the filter base $\mathcal{S}$.

**Proof.** First, observe that the family $\mathcal{B}$ consists of *rescaled* versions of absorbing, balanced, and convex sets. But these properties are invariant under positive scaling and, therefore, like $\mathcal{S}$, also $\mathcal{B}$ consists of absorbing, balanced, and convex sets. Moreover, as pointed out in **Remark 4.4**, $\mathcal{B}$ satisfies condition (4.1). Therefore, according to **Proposition 4.3**, it is sufficient to show that $\mathcal{B}$ is still a filter base on $X$. For that, let $\lambda_1 S_1$ and $\lambda_2 S_2$ be two elements of $\mathcal{B}$ with $S_1, S_2 \in \mathcal{S}$ and $\lambda_1, \lambda_2 > 0$. We set $\lambda := \lambda_1 \wedge \lambda_2$ and consider $S \in \mathcal{S}$ such that $S \subseteq S_1 \cap S_2$ (such an element $S \in \mathcal{S}$ exists because, by assumption, $\mathcal{S}$ is a filter base). But then, we have $\lambda S \in \mathcal{B}$ and $\lambda S \subseteq \lambda_1 S_1 \cap \lambda_2 S_2$ (because $S, S_1, S_2$ are balanced sets). ■■■■

**4.7. Remark.** Note that the family $\mathcal{B}_\sharp := \bigcup_{n \in \mathbb{N}^*} (1/n)\mathcal{S}$ is also a filter base of neighborhoods of the origin for the same topology generated by $\mathcal{B} := \bigcup_{\lambda \in \mathbb{R}_+^*} \lambda \mathcal{S}$; this is a consequence of **Corollary 1.77**. In particular, if $\mathcal{S}$ consists of countably many elements, so does $\mathcal{B}_\sharp$ and, therefore, in this case, the locally convex space generated by $\mathcal{S}$ is first countable. For example, if $X = \mathbb{R}^2$ and $\mathcal{S} = \{\mathbb{D}_\bullet\}$ then the locally convex space generated by $\mathcal{S}$ is the standard Euclidean topology.

However, note that if $\mathcal{S}$ consists of an uncountable number of elements, then, in general, the locally convex space generated by $\mathcal{S}$ is not first countable (it depends on the possibility to find a countable filter base equivalent to the uncountable filter base $\mathcal{S}$). For example, if $X = \mathbb{R}^2$ and $\mathcal{S} = \bigcup_{\lambda \in \mathbb{R}_+^*} \{\lambda \mathbb{D}_\bullet\}$ then $\mathcal{S}$ is uncountable. But still, the countable family $\mathcal{S}_\sharp := \bigcup_{n \in \mathbb{N}^*} \{(1/n)\mathbb{D}_\bullet\}$ is filter base on $\mathbb{R}^2$ equivalent to $\mathcal{S}$ which, again, generates the standard Euclidean topology.

**Preview**. Let $X$ be a (pure algebraic) vector space. The procedure to construct locally convex spaces goes as follows:

1. Construct a **filter base** $\mathcal{S}$ on $X$ consisting of sets that are at the same time absorbing, balanced, and convex.

2. Then, construct $\mathcal{B} := \cup_{\lambda \in \mathbb{R}_+^*} \lambda \mathcal{S}$. The family $\mathcal{B}$ is still a filter base on $X$ and satisfies the characterization property stated in Proposition 4.3:

$$\forall U \in \mathcal{B},\ \forall \rho \in \mathbb{R}_+^*, \quad \exists W \in \mathcal{B} :: W \subseteq \rho U.$$

The previous property assures that the filter base $\mathcal{B}$ is a fundamental system of neighborhoods of the origin for a locally convex topology on $X$.

Step 2 in the previous procedure is easy and mechanical. By contrast, the first step can be demanding, and one would like to find more natural ways to construct a filter base $\mathcal{S}$ on $X$ consisting of absorbing, balanced, and convex sets. This is the next section's main aim, where we introduce the so-called **filtering** families of seminorms and show how they allow for a simple way to construct $\mathcal{S}$. Indeed, Lemma 4.14 below permits to replace the first step with the following two substeps:

1.1. Construct a **filtering** family of seminorms $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ on the (**purely**) **algebraic** vector space $X$.

1.2. Set $\mathcal{S} := \{B_\bullet(\mathfrak{p}_\alpha)\}_{\alpha \in \mathcal{A}}$. By Lemma 4.14, $\mathcal{S}$ is a filter base on $X$ (consisting of convex, absorbing, and balanced).

Eventually, Proposition 4.17 below explains how it is always possible to extend a family of seminorms $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ so that the resulting family $(\mathfrak{p}_\gamma)_{\gamma \in \Gamma}$ is filtering (and $(\mathfrak{p}_\gamma)_{\gamma \in \Gamma} \supseteq (\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$). ...

### 4.1.1. Locally convex spaces defined by a family of seminorms

**4.8. Definition.** Let $X$ be a vector space and $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ a family of seminorms on $X$. We say that the family $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ is **total** (or **separating** or that it **separates the points**) if for every $0 \neq x \in X$, **different from the zero vector**, there exists an $\alpha \in \mathcal{A}$, depending on $x$, such that $\mathfrak{p}_\alpha(x) \neq 0$. Equivalently, the family $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ separate the points if, whenever $\mathfrak{p}_\alpha(x) = 0$ holds for every $\alpha \in \mathcal{A}$, then necessarily $x = 0$.

**4.9. Remark.** The motivation to introduce families of seminorms on a vector space $X$ is that if $\mathfrak{p}$ is a seminorm on $X$, then both the open and closed unit semiballs $B_\circ(\mathfrak{p})$ and $B_\bullet(\mathfrak{p})$ can have a shape that differs from the intuitive idea of a ball. Indeed, both $B_\circ(\mathfrak{p})$ and $B_\bullet(\mathfrak{p})$ contain the set of points of $X$ where $\mathfrak{p}$ vanishes, i.e., the kernel of $\mathfrak{p}$ that we know to be a nontrivial vector space if $\mathfrak{p}$ is not a norm. Considering families of seminorms allows fixing this issue. For example, if we consider two seminorms $\mathfrak{p}$ and $\mathfrak{q}$ such that $(\ker \mathfrak{p}) \cap (\ker \mathfrak{q}) = \{0\}$, then the intersection $B_\circ(\mathfrak{p}) \cap B_\circ(\mathfrak{q})$ has a more familiar shape in the sense that $B_\circ(\mathfrak{p}) \cap B_\circ(\mathfrak{q})$ cannot contain lines. Indeed, if $\lambda x \in B_\circ(\mathfrak{p}) \cap B_\circ(\mathfrak{q})$ for every $\lambda \neq 0$ then $\mathfrak{p}(x) \leqslant |\lambda|^{-1}$ and $\mathfrak{q}(x) \leqslant |\lambda|^{-1}$ for every $\lambda \neq 0$. But this implies $\mathfrak{p}(x) = \mathfrak{q}(x) = 0$, i.e., $x \in (\ker \mathfrak{p}) \cap (\ker \mathfrak{q}) = \{0\}$, i.e., $x = 0$.

**4.10. Remark.** Note that, if $\mathfrak{p}$ is a seminorm on a vector space $X \neq \{0\}$ and $\mathfrak{p}(x) \neq 0$, then the restriction of $\mathfrak{p}$ to the $1$-dimensional subspace $X_1 := \mathrm{span}(x)$ is a norm on $X_1$. Therefore, if $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ is total, then for every $0 \neq x \in X$, there exists a seminorm $\mathfrak{p}_\alpha$ (defined on the whole space $X$), whose restriction $\mathfrak{p}|X_1$ to the 1-dimensional subspace $X_1 := \mathrm{span}(x)$ is a norm on $X_1$.

**4.11. Definition.** Let $X$ be a vector space and $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ a family of seminorms on $X$. We say that the family $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ is **directed**, or **filtering**, if the **ordered set** $(\{\mathfrak{p}_\alpha\}_{\alpha \in \mathcal{A}}, \preccurlyeq)$ is a **directed set**. Here, with $\preccurlyeq$ we mean the usual order relation defined by

$$p_\alpha \preccurlyeq p_\beta \quad \textbf{if and only if} \quad p_\alpha(x) \leqslant p_\beta(x) \quad \forall x \in X,$$

Note that the index set $\mathcal{A}$ is **not** assumed to be directed.

**4.12. Remark.** More explicitly, the family $(\mathfrak{p}_\alpha)_{\alpha\in\mathcal{A}}$ is directed if, and only if, for every pair of seminorms $\mathfrak{p}_{\alpha_1}$ and $\mathfrak{p}_{\alpha_2}$ there always exists a seminorm $\mathfrak{p}_\alpha$ upper bounding them, that is, such that $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_{\alpha_1}$ and $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_{\alpha_2}$ (equivalently, such that $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_{\alpha_1} \vee \mathfrak{p}_{\alpha_2}$).

**Recall the Notation 3.60:** If $\mathfrak{p}$ is a continuous seminorm on the topological vector space $\mathfrak{X}$ there is no need anymore to distinguish $B_\bullet$ and $\overline{B_\circ}$, or $B_\circ$ and $(B_\bullet)^\circ$. But still, one can work with two (or even a family of) continuous seminorms defined in $\mathfrak{X}$, and it becomes essential to distinguish among different unit semiballs. Therefore, we shall also denote the open unit semiball of $\mathfrak{p}$ by $B_\circ(\mathfrak{p})$ and the closed unit semiball of $\mathfrak{p}$ by $B_\bullet(\mathfrak{p})$.

**4.13. Remark.** Recall the result stated in **Proposition 1.49**. If $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_\beta$ then $B_\bullet(\mathfrak{p}_\alpha) \subseteq B_\bullet(\mathfrak{p}_\beta)$ and vice versa. In symbols: $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_\beta$ if, and only if, $B_\bullet(\mathfrak{p}_\alpha) \subseteq B_\bullet(\mathfrak{p}_\beta)$. ⋯

The following result motivates the term *filtering* given to such families of seminorms.

**4.14. Lemma.** *Let $(\mathfrak{p}_\alpha)_{\alpha\in\mathcal{A}}$ be a* **filtering** *family of seminorms on a* (**purely**) **algebraic** *vector space $X$. The family $\mathcal{S} := \{B_\bullet(\mathfrak{p}_\alpha)\}_{\alpha\in\mathcal{A}}$ is a filter base on $X$ (consisting of absorbing, balanced, and convex sets).*

Note that we are not stating that $\mathcal{S}$ is a filter base *of neighborhoods* of the origin (and indeed it would make no sense because, up to now, there is no topology on $X$) but just a filter base in the purely set theoretical sense.

**Proof.** First note that $\emptyset \notin \mathcal{S}$ as $0 \in B_\bullet(\mathfrak{p}_\alpha)$ for every $\alpha \in A$. Next, let $B_\bullet(\mathfrak{p}_{\alpha_1})$ and $B_\bullet(\mathfrak{p}_{\alpha_2})$ be two closed unit semiballs (associated to the seminorms $\mathfrak{p}_{\alpha_1}$ and $\mathfrak{p}_{\alpha_2}$). By hypothesis, there exists a seminorm $\mathfrak{p}_\alpha$ such that $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_{\alpha_1}$ and $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_{\alpha_2}$. But, for any $\alpha, \beta \in \mathcal{A}$

$$\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_\beta \quad \textbf{implies} \quad B_\bullet(\mathfrak{p}_\alpha) \subseteq B_\bullet(\mathfrak{p}_\beta).$$

Hence, $B_\bullet(\mathfrak{p}_\alpha) \subseteq B_\bullet(\mathfrak{p}_{\alpha_1}) \cap B_\bullet(\mathfrak{p}_{\alpha_2})$, and this shows that $\mathcal{S}$ is a filter base on $X$ (cf. **Proposition 1.74**). ■

**Exercise**. Let $(\mathfrak{p}_\alpha)_{\alpha\in\mathcal{A}}$ be a family of seminorms. Prove the converse of **Proposition 4.14**, that is, if the family $\mathcal{S} := \{B_\bullet(\mathfrak{p}_\alpha)\}_{\alpha\in\mathcal{A}}$ is a filter base on $X$, then $(\mathfrak{p}_\alpha)_{\alpha\in\mathcal{A}}$ is a filtering family of seminorms. **Solution.** Consider two seminorms $\mathfrak{p}_{\alpha_1}, \mathfrak{p}_{\alpha_2}$ of the family, and let $B_\bullet(\mathfrak{p}_{\alpha_1}), B_\bullet(\mathfrak{p}_{\alpha_2})$ be the corresponding closed unit semiballs. Since $\mathcal{S}$ is a filter base on $X$, then there exists a closed semiball $B(\mathfrak{p}_\alpha)$ such that $B(\mathfrak{p}_\alpha) \subseteq B_\bullet(\mathfrak{p}_{\alpha_1}) \cap B_\bullet(\mathfrak{p}_{\alpha_2})$. By **Remark 4.13**, it follows that $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_{\alpha_1} \vee \mathfrak{p}_{\alpha_2}$ and we conclude.

**4.15. Proposition. Assumption:** *Let $(\mathfrak{p}_\alpha)_{\alpha\in\mathcal{A}}$ be a* **filtering** *family of seminorms on a* (**purely**) **algebraic** *vector space $X$. Then:*

Continuitiy of the seminorms make no sense in this purely algebraic context

**Claim *i*.** *It is possible to structure $X$ into a locally convex space $\mathfrak{X} := (X, \tau_{\mathrm{LC}})$ by declaring as a fundamental system of neighborhoods of the origin the set consisting of all possible closed semiballs (of any strictly positive «radius») of the seminorms of the family. In other words, we define a fundamental system of neighborhoods of the origin of $X$ by setting*

$$\mathcal{B} := \{\rho B_\bullet(\mathfrak{p}_\alpha)\}_{(\rho,\alpha)\in\mathbb{R}_+^*\times\mathcal{A}}.$$

*In the framework of* **Proposition 4.5**, *this corresponds to the choice $\mathcal{S} := \{B_\bullet(\mathfrak{p}_\alpha)\}_{\alpha\in\mathcal{A}}$.*

**Claim *ii*.** *Every seminorm $\mathfrak{p}_\alpha$ is then* **continuous** *on $X$ with respect to this topology $\tau_{\mathrm{LC}}$ (generated by $\mathcal{B}$) and therefore, for every $\alpha \in \mathcal{A}$ we have*

$$(B_\bullet(\mathfrak{p}_\alpha))^\circ = B_\circ(\mathfrak{p}_\alpha) \quad \textbf{and} \quad B_\bullet(\mathfrak{p}_\alpha) = \overline{B_\circ(\mathfrak{p}_\alpha)}.$$

Now it makes sense to talk about continuous seminorms (continuous with respect to the $\tau_{\mathrm{LC}}$ topology just defined)

**Claim *iii*.** *The locally convex topology $\tau_{\mathrm{LC}}$ is (Hausdorff) separated if, and only if, the family $(\mathfrak{p}_\alpha)_{\alpha\in\mathcal{A}}$ separates the points.*

**4.16. Definition.** If the locally convex topology of a topological vector space $\mathfrak{X}$ is constructed as described in **Proposition 4.15**, we say that **the locally convex topology of $\mathfrak{X}$ is defined by the filtering family of seminorms** $(\mathfrak{p}_\alpha)_{\alpha\in\mathcal{A}}$. If this is the case, every seminorm $\mathfrak{p}_\alpha$, $\alpha\in\mathcal{A}$, is necessarily continuous on $\mathfrak{X}$. Moreover, $(\mathfrak{p}_\alpha)_{\alpha\in\mathcal{A}}$ is **total** if, and only if, the topology of $\mathfrak{X}$ is **Hausdorff separated**.

Recall that "total" is a synonym of "separates the points" and of "separating".

PROOF. ***i.*** Let $B_\bullet(\mathfrak{p}_\alpha)$ be the **closed** unit semiball of the seminorm $\mathfrak{p}_\alpha$. We set $\mathcal{S}:=\{B_\bullet(\mathfrak{p}_\alpha)\}_{\alpha\in\mathcal{A}}$ and $\mathcal{B}:=\cup_{\rho\in\mathbb{R}_+^*}\rho\mathcal{S}$. We show that $\mathcal{B}$ is a filter base of neighborhoods of $0_X$ for a locally convex topology on $X$, which is compatible with the vector space structure of $X$.

For that, as the semiballs are convex, absorbing and balanced sets, it is sufficient to show, according to **Proposition 4.5**, that $\mathcal{S}$ is a filter base on $X$. But this has been shown in **Lemma 4.14**.

***ii.*** From the way the topology is defined, $B_\bullet(\mathfrak{p}_\alpha)$ is a neighborhood of the origin. Therefore (cf. **Proposition 3.61**) $\mathfrak{p}_\alpha$ is continuous. An alternative proof consists in showing (thanks again to the equivalences stated in **Proposition 3.61**) that $\mathfrak{p}_\alpha$ is continuous at the origin $0_X\in X$, i.e., that for every $\varepsilon>0$, there exists $\rho>0$ such that $\mathfrak{p}_\alpha(\rho B_\bullet(\mathfrak{p}_\alpha))<\varepsilon$ (and by this we mean $\mathfrak{p}_\alpha(\rho x)<\varepsilon$ for any $x\in B_\bullet(\mathfrak{p}_\alpha)$). As $\mathfrak{p}_\alpha(\rho B_\bullet(\mathfrak{p}_\alpha))<\rho$, it is sufficient to take $\rho:=\varepsilon$.

***iii.*** Let $x_0\neq 0$. In agreement with **Proposition 3.20**, it is sufficent to show the existence of a neighborhood of the origin that does not pass through $x_0$. For that, we observe that, by hypothesis, there exists $\alpha_0\in\mathcal{A}$ such that $\mathfrak{p}_{\alpha_0}(x_0)\neq 0$. Thus, the closed semiball

$$\frac{\mathfrak{p}_{\alpha_0}(x_0)}{2}B_\bullet(\mathfrak{p}_{\alpha_0})\equiv\left\{x\in X :: \mathfrak{p}_{\alpha_0}(x)<\frac{\mathfrak{p}_{\alpha_0}(x_0)}{2}\right\}$$

does not contain $x_0$. This shows that the locally convex space $(X,\tau_{\mathrm{LC}})$ defined on $X$ by the filtering family of seminorms $(\mathfrak{p}_\alpha)_{\alpha\in\mathcal{A}}$ is (Hausdorff) separated.

On the other hand, suppose $\tau_{\mathrm{LC}}$ Hausdorff separated. Let $0\neq x_0\in X$ be a generic non-zero point of $X$. As $\tau_{\mathrm{LC}}$ is Hausdorff separated, there exist $(\alpha_0,\rho_0)\in\mathcal{A}\times\mathbb{R}_+^*$ such that

$$x_0\notin\rho_0 B_\bullet(\mathfrak{p}_{\alpha_0})\equiv\{x\in X :: \mathfrak{p}_{\alpha_0}(x)<\rho_0\}.$$

As $\ker\mathfrak{p}_{\alpha_0}\subseteq\rho_0 B_\bullet(\mathfrak{p}_{\alpha_0})$ we necessarily have $\mathfrak{p}_{\alpha_0}(x_0)\neq 0$. The arbitrariness of $x_0$ shows that the family $(\mathfrak{p}_\alpha)_{\alpha\in\mathcal{A}}$ is total.

The following result explains how any family of seminorms can be extended to a *filtering* family of seminorms.

**4.17. Proposition. Assumption:** *Let $(\mathfrak{p}_\gamma)_{\gamma\in\Gamma}$ be any family of seminorms on a vector space $X$. The family $(\mathfrak{p}_\gamma)_{\gamma\in\Gamma}$ can be* **filtering** *or* **not**.

**Construction.** *Denote by $\mathcal{A}$ the finite subsets of $\Gamma$, i.e., the subset of $\wp(\Gamma)$ consisting of elements having finite cardinality. For every $[\alpha]\in\mathcal{A}\subseteq\wp(\Gamma)$ set*

$$\mathfrak{p}_{[\alpha]}:=\sup_{\gamma\in[\alpha]}\mathfrak{p}_\gamma.$$

*Note that, for every $x\in\mathfrak{X}$, the* sup *is extended to a finite subset of real numbers and therefore, $\mathfrak{p}_{[\alpha]}(x):=\max_{\gamma\in[\alpha]}\mathfrak{p}_\gamma(x)$.*

**Claim.** *The family $(\mathfrak{p}_{[\alpha]})_{[\alpha]\in\mathcal{A}}$ is a filtering family of seminorms on $X$.*

**Example 4.18.** To get acquainted with the square bracket notation $[\alpha]$, with $[\alpha]\in\mathcal{A}$ and $\mathcal{A}$ the set of all finite subsets of $\Gamma$, let us explicitly write down the formulae of the "construction step" in a specific case.

Consider $\Gamma := \mathbb{N}$, i.e., the case of a sequence of seminorms. Then $[\alpha] \in \mathcal{A} \subseteq \wp(\Gamma)$ can be the set $\mathbb{N}_n := \{1, ..., n\}$ for some $n \in \mathbb{N}$, or any other finite subset of natural numbers. The construction step produces, for example,

$$\mathfrak{p}_{\mathbb{N}_n} := \sup_{\gamma \in \mathbb{N}_n} \mathfrak{p}_\gamma, \quad \textbf{that is} \quad \mathfrak{p}_{\mathbb{N}_n}(x) := \sup_{\gamma \in \mathbb{N}_n} \mathfrak{p}_\gamma(x) \quad \forall x \in X.$$

In other words, the new family is obtained by taking, **pointwise**, the supremum, which is actually a maximum because it is extended to a finite set of real numbers.

**PROOF.** Indeed, let $[\alpha_1]$ and $[\alpha_2]$ be two elements of $\mathcal{A}$. Set $[\alpha] := [\alpha_1] \cup [\alpha_2] \in \mathcal{A}$. Then, with $\mathfrak{p}_{[\alpha_1]} := \sup_{\gamma \in [\alpha_1]} \mathfrak{p}_\gamma$ and $\mathfrak{p}_{[\alpha_2]} := \sup_{\gamma \in [\alpha_2]} \mathfrak{p}_\gamma$, we get

$$\mathfrak{p}_{[\alpha]} := \sup_{\gamma \in [\alpha]} \mathfrak{p}_\gamma = \sup \left\{ \sup_{\gamma \in [\alpha_1]} \mathfrak{p}_\gamma, \sup_{\gamma \in [\alpha_2]} \mathfrak{p}_\gamma \right\} = \sup\{\mathfrak{p}_{[\alpha_1]}, \mathfrak{p}_{[\alpha_2]}\}.$$

Hence, in this way, we build a partially ordered set $((\mathfrak{p}_{[\alpha]})_{[\alpha] \in \mathcal{A}}, \preccurlyeq)$ which turns out to be a join-semilattice as $\mathfrak{p}_{[\alpha_1]} \vee \mathfrak{p}_{[\alpha_2]} = \mathfrak{p}_{[\alpha]} = \mathfrak{p}_{[\alpha_1] \cup [\alpha_2]}$. In particular, it is a directed set and, therefore, by definition, a filtering family of seminorms. ■■■■

**4.19. Remark.** Observe that for any $[\alpha], [\beta] \in \mathcal{A}$ one has $\mathfrak{p}_{[\alpha]} \succcurlyeq \mathfrak{p}_{[\beta]}$ **if, and only if** $B_\bullet(\mathfrak{p}_{[\alpha]}) \subseteq B_\bullet(\mathfrak{p}_{[\beta]})$. Therefore, as $\mathfrak{p}_{[\alpha]} \succcurlyeq \mathfrak{p}_\gamma$ for every $\gamma \in [\alpha]$, we have that

$$B_\bullet(\mathfrak{p}_{[\alpha]}) \subseteq \bigcap\nolimits_{\gamma \in [\alpha]} B_\bullet(\mathfrak{p}_\gamma) \quad \textbf{for any } [\alpha] \in \mathcal{A}.$$

Moreover, as $[\{\gamma\}] \in \mathcal{A}$ for any $\gamma \in \Gamma$ and $\mathfrak{p}_\gamma \equiv \mathfrak{p}_{[\{\gamma\}]}$, we have that $(\mathfrak{p}_{[\alpha]})_{[\alpha] \in \mathcal{A}}$ is an **extension** of the family $(\mathfrak{p}_\gamma)_{\gamma \in \Gamma}$. In other words, $\{\mathfrak{p}_\gamma\}_{\gamma \in \Gamma} \subseteq \{\mathfrak{p}_{[\alpha]}\}_{[\alpha] \in \mathcal{A}}$. But this means that the family $\mathcal{B} = \{\rho B_\bullet(\mathfrak{p}_{[\alpha]})\}_{(\rho, [\alpha]) \in \mathbb{R}_+^* \times \mathcal{A}}$ is a filter base for a locally convex topology on $X$ that includes the family $\{\rho B_\bullet(\mathfrak{p}_\gamma)\}_{(\rho, \gamma) \in \mathbb{R}_+^* \times \Gamma}$ which, in general, *needs not* to be a filter base on $X$. Note that, what we just said, is consistent with **Proposition 4.15**, as the family $(\mathfrak{p}_\gamma)_{\gamma \in \Gamma}$, *not* assumed to be *filtering*, does not fall under the assumptions of **Proposition 4.15**.

∘

∘

### 4.1.2. Characterization of locally convex spaces

This section aims to show that every locally convex space can be defined by a *filtering* family of (*continuous*) seminorms. More precisely, the following characterization holds.

Recall the definition of **barrelled set** (given in **Definition 3.62**): In a topological vector space, we call **barrel** (or **barrelled set**, or **tonneau**) every set which is **absorbing**, **balanced**, **convex** and **closed**.

**4.21. Theorem. Assumption.** *Let $(\mathfrak{X}, \mathcal{V})$ be a locally convex space.*

**Claim *i*.** *There exists a fundamental system of neighborhoods of the origin consisting of barrelled sets.*

**Claim *ii*.** *Consider any fundamental system of neighborhoods of the origin consisting of barrelled sets. At least one such system exists due to* **Claim *i***. *Call it $\mathcal{T}$. Then, the family of* **gauges**

$$(\mathfrak{p}_T)_{T \in \mathcal{T}}$$

*forms a* **filtering** *family of (continuous) seminorms, which defines the locally convex topology of $\mathfrak{X}$.*

**4.22. Remark.** Formally, cf. **Proposition 4.15**, Claim *ii.* states that if we set $\mathcal{S} := \{B_\bullet(\mathfrak{p}_T)\}_{T\in\mathcal{T}}$ then $\mathcal{B} = \bigcup_{\rho>0}\rho\mathcal{S} = \bigcup_{(\rho,T)\in\mathbb{R}_+^*\times\mathcal{T}}\{\rho B_\bullet(\mathfrak{p}_T)\}$ is a fundamental system of neighborhoods of the origin for $(\mathfrak{X},\mathcal{V})$. But in this case, already $\mathcal{S}$ is a fundamental system of neighborhoods of the origin. Therefore, in this specific case, the construction returns a fundamental system of neighborhoods of the origin $\mathcal{B}$ which, in general, is bigger than $\mathcal{S}$.

**Proof.** ***i.*** Let $U\in\mathcal{V}(0)$ be a neighborhood of the origin $0\in\mathfrak{X}$. Since every topological vector space is regular (cf. **Proposition 3.23**), there exists a **closed** neighborhood of the origin $V\in\mathcal{V}(0)$ such that $V\subseteq U$. Since $\mathfrak{X}$ is locally convex, $V$ contains a **convex** neighborhood of the origin $W\in\mathcal{V}(0)$. But $W$ contains a balanced neighborhood $E\in\mathcal{V}(0)$. Summarizing, given $U\in\mathcal{V}(0)$, there exist neighborhoods $V,W,E\in\mathcal{V}(0)$ such that

$$U \supseteq V\ (\text{closed}) \supseteq W\ (\text{convex}) \supseteq E\ (\text{balanced}). \tag{4.2}$$

Next, we consider the convex envelope $K(E)$ of the balanced set $E$. We know (cf. **Corollary 1.34**) that $K(E)$ is still balanced (and convex). Clearly, by (4.2), we get that $\overline{K(E)}\subseteq\bar{W}\subseteq V\subseteq U$. Since the closure of a convex and balanced set is still convex and balanced (cf. **Proposition 3.10**) $\overline{K(E)}$ is a barrelled set (as usual, the fact that $\overline{K(E)}$ is absorbing comes from the fact that every neighborhood is absorbing, and every superset of an absorbing set is still absorbing). The proof of the first claim is completed.

***ii.*** According to **Proposition 3.65**, for any $T\in\mathcal{B}$ the **gauge** $\mathfrak{p}_T$ is a continuous seminorm whose closed unit semiball coincides with $T$: $B_\bullet(\mathfrak{p}_T)=T$. Therefore, the set of all these closed *unit* semiballs $\mathcal{S} := \{B_\bullet(\mathfrak{p}_T)\}_{T\in\mathcal{B}}$ is a filter base of **barrelled** neighborhoods of the origin $0\in\mathfrak{X}$. All the more reason, the set $\mathcal{B} := \cup_{\lambda\in\mathbb{R}_+^*}\lambda\mathcal{S}$ consisting of all closed semiballs (of any positive radius) is a filter base of barrelled neighborhoods of the origin of the locally convex space $\mathfrak{X}$.

Eventually, it is straightforward to check that $\{\mathfrak{p}_T\}_{T\in\mathcal{B}}$ is a **filtering** family of (continuous) seminorms because the intersection of a finite number of barrelled sets is still a barrelled set (cf. **Proposition 1.26**).

The previous result has the following remarkable consequence.

**4.23. Corollary.** *Let $\mathfrak{X}$ be a* **locally convex space***. The set of all* **closed unit** *semiballs generated by the system of all possible continuous seminorms, i.e., the family*

$$\{B_\bullet(\pi)\}_{\pi\in\mathfrak{F}_\mathfrak{X}} \quad \textit{with} \quad \mathfrak{F}_\mathfrak{X} = \{\pi:\mathfrak{X}\to\mathbb{R}_+ :: \pi \textbf{ is a continuous seminorm on } \mathfrak{X}\},$$

*is a fundamental system of neighborhoods of the origin for $\mathfrak{X}$.*

**Proof.** Indeed $\{B_\bullet(\pi)\}_{\pi\in\mathfrak{F}_\mathfrak{X}}$ is nothing but the system $\mathcal{T}$ consisting of all barrelled neighborhoods of the origin (which we just showed to be a fundamental system of neighborhoods in a locally convex space). To see this, we observe that $B_\bullet(\pi)$, being the closed unit semiball of a continuous seminorm is a barrelled neighborhood of the origin. Therefore $\{B_\bullet(\pi)\}_{\pi\in\mathfrak{F}_\mathfrak{X}}\subseteq\mathcal{T}$. On the other hand, if $T\in\mathcal{T}$ then (cf. **Proposition 3.67**) the gauge $\mathfrak{p}_T$ is a continuous seminorm whose closed unit semiball coincides with $T$: $B_\bullet(\mathfrak{p}_T)=T$. Hence, $\mathcal{T}\subseteq\{B_\bullet(\pi)\}_{\pi\in\mathfrak{F}_\mathfrak{X}}$.

**4.24. Remark.** Note that the family $\mathfrak{F}_\mathfrak{X}$ is a **filtering** family of seminorms. More precisely, the family $\{B_\bullet(\pi)\}_{\pi\in\mathfrak{F}_\mathfrak{X}}$ forms a join-semilattice. Indeed, in **Example 1.39**, we have shown that the join of two seminorms is still a seminorm. But then, the assertion follows from the fact that the join of two continuous real-valued functions is still a continuous function.

**4.25. Remark.** Note that a vector space $X$ endowed with the indiscrete topology $\{\emptyset, X\}$ is a locally convex space. The unique continuous seminorm on such a trivial space is the trivial seminorm, i.e., the seminorm identically equal to zero. This because $\mathfrak{p} \equiv 0$ is the only constant seminorm (in fact, in general, if $f: X \to Y$ is a map between the topological spaces $X$ and $Y$, where $X$ is endowed with the indiscrete topology, and $Y$ is Hausdorff separated, then $f$ is continuous if, and only if, $f$ is constant). Apart from this trivial case in which the set $\mathfrak{F}_{\mathfrak{X}}$ reduces to just one element (the null seminorm), in general, the family $\mathfrak{F}_{\mathfrak{X}}$ is uncountable. Indeed, in general, $\mathfrak{F}_{\mathfrak{X}}$ is a positive cone, because if $\mathfrak{p}, \mathfrak{q} \in \mathfrak{F}_{\mathfrak{X}}$ then also $\mathfrak{p} + \lambda\mathfrak{q} \in \mathfrak{F}_{\mathfrak{X}}$ whenever $\lambda \geqslant 0$.

## 4.2 Bases of continuous seminorms

Recall the notation for the set of all possible continuous seminorms on $\mathfrak{X}$:

$$\mathfrak{F}_{\mathfrak{X}} = \{\pi \colon \mathfrak{X} \to \mathbb{R}_+ :: \pi \textbf{ is a continuous seminorm on } \mathfrak{X}\}.$$

We have the following useful characterization.

**4.26. Definition.** Let $\mathfrak{B}_{\mathfrak{X}} \subseteq \mathfrak{F}_{\mathfrak{X}}$ be a family (filtering or not) of *continuous* seminorms all defined on the same *locally convex* topological vector space $\mathfrak{X}$. We say that $\mathfrak{B}_{\mathfrak{X}}$ is a **basis of continuous seminorms** (or **a fundamental system of continuous seminorms**) on $\mathfrak{X}$ if the set consisting of all *closed* semiballs (of any strictly positive «radius»)

$$\{\rho\, B_\bullet(\mathfrak{p})\}_{(\rho,\mathfrak{p}) \in \mathbb{R}_+^* \times \mathfrak{B}_{\mathfrak{X}}} \tag{4.3}$$

is *a fundamental system of neighborhoods of the origin* in $\mathfrak{X}$. If $\mathfrak{B}_{\mathfrak{X}}$ is a basis of continuous seminorms, to emphasize, we sometimes refer to its elements as *basic* seminorms.

**4.27. Remark.** Note that, we do not ask that $\{B_\bullet(\mathfrak{p})\}_{\mathfrak{p} \in \mathfrak{B}_{\mathfrak{X}}}$ is a fundamental system of neighborhoods of the origin, but only that $\{\rho\, B_\bullet(\mathfrak{p})\}_{(\rho,\mathfrak{p}) \in \mathbb{R}_+^* \times \mathfrak{B}_{\mathfrak{X}}}$ in (4.3) is a fundamental system of neighborhoods of the origin.

By the very definition, a filtering family of seminorms $\mathfrak{P}_{\mathfrak{X}}$ that defines a locally convex space $\mathfrak{X}$, consists (a posteriori) of continuous seminorms. A posteriori, $\mathfrak{P}_{\mathfrak{X}}$ is also a basis of continuous seminorms (cf. **Proposition 4.15**, **Claim *i***). Reciprocally, **Theorem 4.21** and **Corollary 4.23**, can be equivalently stated under the form that *every locally convex space admits a basis of continuous seminorms* (in fact, even a filtering family of seminorms).

**4.28. Proposition.** *Let $\mathfrak{X}$ be a* **locally convex space** *and $\mathfrak{P} \subseteq \mathfrak{F}_{\mathfrak{X}}$ a family of* continuous *seminorms on $\mathfrak{X}$. The following assertions are equivalent:*

*i.* *The family $\mathfrak{P}$ is a* **basis** *of continuous seminorms.*

*ii.* *For every continuous seminorm $\pi \in \mathfrak{F}_{\mathfrak{X}}$, there exists a basic seminorm $\mathfrak{p} \in \mathfrak{P}$ and a constant $c_\pi > 0$ such that (both $\mathfrak{p}$ and $c_\pi$ can depend on $\pi$)*

$$\pi(x) \leqslant c_\pi \mathfrak{p}(x) \quad \forall x \in \mathfrak{X}.$$

*Statement ii. justifies the name «basis of continuous seminorms» as such a subfamily of $\mathfrak{F}_{\mathfrak{X}}$ plays the role of a generator for all possible continuous seminorms on $\mathfrak{X}$* (cf. **Corollary 4.23**).

**4.29. Remark.** In other words, condition ***ii.*** tells us that $\mathfrak{P}$ is a basis of continuous seminorms if, and only if, every element of the set $\mathfrak{F}_{\mathfrak{X}}$ of all continuous seminorms on $\mathfrak{X}$ can be upper bounded by *some* element of $\mathfrak{P}$ (up to a constant positive factor).

**4.30. Remark.** Note that if $\mathfrak{P}$ consists of a single seminorm $\mathfrak{p}$, then $\mathfrak{P}$ is a basis of continuous seminorms if, and only if, for every $\pi \in \mathfrak{F}_{\mathfrak{X}}$ there exists $c_\pi > 0$ such that $\pi \preccurlyeq c_\pi \mathfrak{p}$. A locally convex space is called *seminormable* if it admits a basis of continuous seminorms consisting of just one seminorm. If that unique seminorm is a norm, we say that the locally convex space is *normable* (cf. **Section 4.3.3**).

**Proof.** We observe that assertion ***ii.*** is equivalent to the following one:

***iii.*** *Given any continuous seminorm $\pi$ on $\mathfrak{X}$, there exists a seminorm $\mathfrak{p}\in\mathfrak{P}$ and a positive constant $c_\pi>0$ such that*

$$c_\pi^{-1}B_\bullet(\mathfrak{p})\subseteq B_\bullet(\pi).$$

The easiest way to show the equivalence is to note that if $\mathfrak{p}$ is a seminorm on $\mathfrak{X}$, then for any $c_\pi>0$ also $c_\pi\mathfrak{p}$ is a seminorm on $\mathfrak{X}$ and we already know (cf. **Remark 4.13**) that $\pi\preccurlyeq c_\pi\mathfrak{p}$ if, and only if, $B_\bullet(c_\pi\mathfrak{p})\subseteq B_\bullet(\pi)$. To conclude, one observes that $B_\bullet(c_\pi\mathfrak{p})=c_\pi^{-1}B_\bullet(\mathfrak{p})$.

It is, therefore, sufficient to show that ***i.*** and ***iii.*** are equivalent.

[***i.***$\Rightarrow$***iii.***] Indeed, $B_\bullet(\pi)$ is a neighborhood of the origin because $\pi$ is a *continuous* seminorm (cf. **Proposition 3.61**). Since $\{\rho B_\bullet(\mathfrak{p})\}_{(\rho,\mathfrak{p})\in\mathbb{R}_+^*\times\mathfrak{P}}$ is a filter basis of neighborhoods of the origin, $B_\bullet(\pi)$ must necessarily contain some $\rho B_\bullet(\mathfrak{p})$ for some $\rho>0$. Therefore, it is sufficient to set $c_\pi:=\rho^{-1}$.

[***iii.***$\Rightarrow$***i.***] We have already proved that the family $\mathcal{B}$ consisting of all closed *unit* semiballs, generated by all possible continuous seminorms, is a filter base of neighborhoods of the origin (cf. **Corollary 4.23**). Now, by assumption, every $B_\bullet(\pi)\in\mathcal{B}$ contains at least a $c_\pi^{-1}B_\bullet(\mathfrak{p})$ for some $c_\pi>0$. But this means that the family $\{\rho B_\bullet(\mathfrak{p})\}_{(\rho,\mathfrak{p})\in\mathbb{R}_+^*\times\mathfrak{P}}$ is a filter base of neighborhoods of the origin. ■■■■

The following result points out that if we already have a basis of continuous seminorms at our disposal (say a family $\mathfrak{B}$ strictly contained in $\mathfrak{F}_{\mathfrak{X}}$) then one can check if another family $\mathfrak{P}\subseteq\mathfrak{F}_{\mathfrak{X}}$ is a basis of continuous seminorms, just by comparing the seminorm in $\mathfrak{P}$ with the seminorms in $\mathfrak{B}$.

**4.31. Corollary.** *Let $\mathfrak{X}$ be a* **locally convex space**, *$\mathfrak{B}$ a* **basis** of continuous seminorms *on $\mathfrak{X}$ and $\mathfrak{P}\subseteq\mathfrak{F}$ a family of* continuous *seminorms on $\mathfrak{X}$. The following assertions are equivalent:*

*i. The family $\mathfrak{P}$ is a* **basis** *of continuous seminorms.*

*ii. For every* basic *continuous seminorm $\mathfrak{q}\in\mathfrak{B}$ on $\mathfrak{X}$, there exists a continuous seminorm $\mathfrak{p}\in\mathfrak{P}$ and a constant $c_{\mathfrak{q}}>0$ such that*

$$\mathfrak{q}(x)\leqslant c_{\mathfrak{q}}\,\mathfrak{p}(x)\quad\forall x\in\mathfrak{X}.\tag{4.4}$$

*The key point here is that we are not more asking (as in* **Proposition 4.28**) *that (4.4) holds for every continuous seminorm $\pi\in\mathfrak{F}$, but just for any $\mathfrak{q}\in\mathfrak{B}$ with $\mathfrak{B}\subseteq\mathfrak{F}$ a* **basis** *of continuous seminorms.*

○

**4.33. Remark.** Let us stress the difference with what we stated in **Proposition 4.28**. Here we already have a **basis** $\mathfrak{B}$ of continuous seminorms on $\mathfrak{X}$ (and $\mathfrak{B}$ can have any cardinality), and we want to check if another family of continuous seminorms $\mathfrak{P}\subseteq\mathfrak{F}$ (for example, a countable one) is a **basis** of continuous seminorms on $\mathfrak{X}$ as well. This result is useful because the family $\mathfrak{F}$ of all possible continuous seminorms on $\mathfrak{X}$ (introduced in **Corollary 4.23**) is an uncountable set, and one does not want to check whether $\mathfrak{P}$ is a basis of continuous seminorms by testing if its elements dominate *all possible* continuous seminorms (see **Proposition 6.3** for an implementation of such a useful simplification).

**Proof.** [***i.***$\Rightarrow$***ii.***] By assumption, both $\mathfrak{B}$ and $\mathfrak{P}$ are bases of continuous seminorms. But then, according to **Proposition 4.28**, for every (basic) continuous seminorm $\mathfrak{q}\in\mathfrak{B}$, there exists a seminorm

$\mathfrak{p}\in\mathfrak{P}$ and a constant $c_{\mathfrak{q}}>0$ such that

$$\mathfrak{q}(x)\leqslant c_{\mathfrak{q}}\,\mathfrak{p}(x). \tag{4.5}$$

for every $x\in\mathfrak{X}$. This proves the implication.

[*ii.*⇒*i.*] The common assumption of the statement is that $\mathfrak{B}$ is a basis of continuous seminorms. Hence, given a continuous seminorm $\pi$ on $\mathfrak{X}$, there exists a seminorm $\mathfrak{q}\in\mathfrak{B}$ and a constant $c_\pi>0$ such that

$$\pi(x)\leqslant c_\pi\mathfrak{q}(x)\quad\forall x\in\mathfrak{X}.$$

On the other hand, assumption *ii.* ensures that, since $\mathfrak{q}\in\mathfrak{B}$, there exists a seminorm $\mathfrak{p}\in\mathfrak{P}$ and a constant $c_{\mathfrak{q}}>0$ such that

$$\mathfrak{q}(x)\leqslant c_{\mathfrak{q}}\,\mathfrak{p}(x)\quad\forall x\in\mathfrak{X}.$$

Hence, $\pi(x)\leqslant c_\pi c_{\mathfrak{q}}\,\mathfrak{p}(x)$ for every $x\in\mathfrak{X}$. The arbitrariness of $\pi\in\mathfrak{F}$ concludes the proof.

### 4.2.1. Characterizing convergence in locally convex spaces

The next result characterizes convergence of generalized sequences in locally convex spaces. Roughly speaking, bases of continuous seminorms suffice.

**4.34. Proposition. Assumptions:** *Let $\mathfrak{B}\subseteq\mathfrak{F}$ be a* **basis** *of* **continuous** *seminorms for a* **locally convex space** $\mathfrak{X}$ *(Hausdorff separated or not) and let $(x_\lambda)_{\lambda\in\Lambda}$ be a generalized sequence in $\mathfrak{X}$.*
**Claim***: The following two assertions are equivalent:*

- ***i.*** $\lim_\Lambda x_\lambda\ni x$;
- ***ii.*** $\lim_\Lambda \mathfrak{p}(x-x_\lambda)=0$ *for every* $\mathfrak{p}\in\mathfrak{B}$.

*Note that in claim* ***ii.*** *we are allowed to use the «equality sign» as the family $\mathfrak{p}(x-x_\lambda)$ takes values in the Hausdorff space $\mathbb{R}$.*

**PROOF.** We can confine ourselves to the case $x=0$. We denote by $\mathcal{B}$ the filter base of neighborhoods of the origin defined in (4.3). Then, $\lim_\Lambda x_\lambda\ni 0$ if, and only if,

$$\forall V\in\mathcal{B},\;\exists\lambda_0\in\Lambda :: x_\lambda\in V\;\textbf{whenever}\;\lambda\succcurlyeq\lambda_0,$$

and this is equivalent to

$$\forall\varepsilon>0,\;\forall\mathfrak{p}\in\mathfrak{B},\;\exists\lambda_0\in\Lambda :: x_\lambda\in\varepsilon B_\bullet(\mathfrak{p})\;\textbf{whenever}\;\lambda\succcurlyeq\lambda_0.$$

To conclude the proof, we simply note that $x_\lambda\in\varepsilon B_\bullet(\mathfrak{p})$ if, and only if, $\mathfrak{p}(x_\lambda)\leqslant\varepsilon$.

**4.35. Remark.** Let us suppose that the locally convex topology on $\mathfrak{X}$ is defined by *making filtering* a family of (continuous) seminorms $(\mathfrak{p}_\gamma)_{\gamma\in\Gamma}$. In other words, suppose that $\mathfrak{X}$ is defined by the filtering family $(\mathfrak{p}_{[\alpha]})_{[\alpha]\in\mathcal{A}}$ constructed in **Proposition 4.17** where $\mathcal{A}$ is the set of all finite subsets of $\Gamma$. Being $\mathfrak{p}_{[\alpha]}:=\max_{\gamma\in[\alpha]}\{\mathfrak{p}_\gamma\}$, we get that

$$\begin{array}{ccc}\lim\limits_\Lambda x_\lambda\ni x & & \lim\limits_\Lambda \mathfrak{p}_\gamma(x-x_\lambda)=0\quad\forall\gamma\in\Gamma\\ \Updownarrow & & \Updownarrow\\ \lim\limits_\Lambda \mathfrak{p}_{[\alpha]}(x-x_\lambda)=0\quad\forall[\alpha]\in\mathcal{A} & \Leftrightarrow & \lim\limits_\Lambda \mathfrak{p}_\gamma(x-x_\lambda)=0\quad\forall\gamma\in[\alpha]\in\mathcal{A}.\end{array}$$

Overall, we have that $\lim_\Lambda x_\lambda \ni x$ if, and only if, $\lim_\Lambda \mathfrak{p}_\gamma(x - x_\lambda) = 0$ for every $\gamma \in \Gamma$. However, note that if $(\mathfrak{p}_\gamma)_{\gamma\in\Gamma}$ is not filtering, then it is not a basis of continuous seminorms. Therefore, the converse of **Proposition 4.34** does not hold (i.e., it is not true that if *i.* and *ii.* hold, then $(\mathfrak{p}_\gamma)_{\gamma\in\Gamma}$ is a basis of continuous seminorms).

### 4.2.2. Characterization of linear and continuous maps among locally convex spaces.

**4.36. Lemma.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two* **locally convex** *spaces and $T$ a* **linear** *map from $\mathfrak{X}$ into $\mathfrak{Y}$. The following two assertions are equivalent:*

*i.* *$T: \mathfrak{X} \to \mathfrak{Y}$ is continuous;*

*ii.* *For every continuous seminorm $\mathfrak{q}$ on $\mathfrak{Y}$, there exists a continuous seminorm $\mathfrak{p}_*$ on $\mathfrak{X}$ such that*

$$|Tx|_{\mathfrak{q}} \leqslant |x|_{\mathfrak{p}_*} \quad \forall x \in \mathfrak{X}. \tag{4.6}$$

*Recall that here $|Tx|_{\mathfrak{q}}$ stands for $\mathfrak{q}(Tx)$ and $|x|_{\mathfrak{p}_*}$ stands for $\mathfrak{p}_*(x)$.*

**Proof.** ***ii.* ⇒ *i.*** It is sufficient to show that $T$ is continuous at $0 \in \mathfrak{X}$, i.e., that $(Tx_\lambda)_{\lambda\in\Lambda} \to 0$ for every generalized sequence $(x_\lambda)_{\lambda\in\Lambda}$ in $\mathfrak{X}$ converging to $0 \in \mathfrak{X}$. For that, due to **Proposition 4.34**, it is sufficient to show that $|Tx_\lambda|_{\mathfrak{q}} \to 0$ in $\mathbb{R}$ for every continuous seminorm $\mathfrak{q} \in \mathfrak{F}_{\mathfrak{Y}}$, given that $|x_\lambda|_{\mathfrak{p}} \to 0$ in $\mathbb{R}$ for every continuous seminorm $\mathfrak{p} \in \mathfrak{F}_{\mathfrak{X}}$. But now, given a generic $\mathfrak{q} \in \mathfrak{F}_{\mathfrak{Y}}$, since $|x_\lambda|_{\mathfrak{p}} \to 0$ in $\mathbb{R}$ for every $\mathfrak{p} \in \mathfrak{F}_{\mathfrak{X}}$, we have, in particular, $|x|_{\mathfrak{p}_*} \to 0$ where $\mathfrak{p}_*$ is the seminorm associated with $\mathfrak{q}$ in the assertion ***ii.*** Therefore, from (4.6), also $|Tx|_{\mathfrak{q}} \to 0$.

***i.* ⇒ *ii.*** Recall **Corollary 4.23**: The set of closed unit balls generated by the system of all possible continuous seminorms, i.e., the family $\{B_\bullet(\pi)\}_{\pi\in\mathfrak{F}_{\mathfrak{X}}}$ with $\mathfrak{F}_{\mathfrak{X}} = \{\pi: \mathfrak{X} \to \mathbb{R}_+ :: \pi$ **is a continuous seminorm**$\}$ is a fundamental system of barrelled neighborhoods of the origin for $\mathfrak{X}$. Let $B_\bullet(\mathfrak{q})$ be the closed unit semiball of the continuous seminorm $\mathfrak{q}$ on $\mathfrak{Y}$. The continúity of $T$ shows that there exists a closed unit ball $B_\bullet(\mathfrak{p})$ associated with a continuous seminorm $\mathfrak{p}$ on $\mathfrak{X}$ such that

$$T(B_\bullet(\mathfrak{p})) \subseteq B_\bullet(\mathfrak{q}).$$

In other words, $|x|_{\mathfrak{p}} \leqslant 1$ implies $|Tx|_{\mathfrak{q}} \leqslant 1$. In particular, it follows that $|x|_{\mathfrak{p}} = 1$ implies $|Tx|_{\mathfrak{q}} \leqslant 1$. Hence, $\sup_{|x|_{\mathfrak{p}}=1} |Tx|_{\mathfrak{q}} \leqslant 1$. **Note** that, for every $\rho > 0$, the expression

$$\sup_{|x|_{\mathfrak{p}}=\rho} \frac{|Tx|_{\mathfrak{q}}}{|x|_{\mathfrak{p}}}$$

does not depend on $\rho$, because if $x = \rho z$ then $\frac{|Tx|_{\mathfrak{q}}}{|x|_{\mathfrak{p}}} = \frac{|Tz|_{\mathfrak{q}}}{|z|_{\mathfrak{p}}}$. Therefore, for every $\rho > 0$ we have

$$\begin{aligned} \sup_{|x|_{\mathfrak{p}}\neq 0} \frac{|Tx|_{\mathfrak{q}}}{|x|_{\mathfrak{p}}} &= \sup_{\rho>0} \sup_{|x|_{\mathfrak{p}}=\rho} \frac{|Tx|_{\mathfrak{q}}}{|x|_{\mathfrak{p}}} \\ &= \sup_{|x|_{\mathfrak{p}}=1} |Tx|_{\mathfrak{q}} \leqslant 1, \end{aligned}$$

so that $|Tx|_{\mathfrak{q}} \leqslant |x|_{\mathfrak{p}}$ if $|x|_{\mathfrak{p}} \neq 0$. **It remains** to consider the case $|x|_{\mathfrak{p}} = 0$, i.e., that if $|x|_{\mathfrak{p}} = 0$ then $|Tx|_{\mathfrak{q}} = 0$. We argue as follows. Let $x \in \mathfrak{X}$ be such that $|x|_{\mathfrak{p}} = 0$. If $\ker \mathfrak{p} = \{0\}$ then $x = 0$ and, therefore, by linearity, $|Tx|_{\mathfrak{q}} = |T0|_{\mathfrak{q}} = |0|_{\mathfrak{q}} = 0$. Instead, if $\ker \mathfrak{p} \neq \{0\}$ and $x$ is a nonzero element of $\ker \mathfrak{p}$, then $\lambda x \in \ker \mathfrak{p}$ for every $\lambda > 0$; in particular, $|\lambda x|_{\mathfrak{p}} \leqslant 1$ for every $\lambda > 0$. But then $|T(\lambda x)|_{\mathfrak{q}} \leqslant 1$ for every $\lambda > 0$, i.e.,

$$|Tx|_{\mathfrak{q}} \leqslant \frac{1}{\lambda} \quad \forall \lambda > 0.$$

But it is not so easy like if $\mathfrak{p}$ were a norm, because here the fact that $|x|_{\mathfrak{p}} = 0$ does not impliy that $x = 0$. Otherwise one simply get $Tx = 0$ by linearity, and therefore $|Tx|_{\mathfrak{q}} = 0$.

Taking the limit for $\lambda \to +\infty$, we get $|Tx|_{\mathfrak{q}} = 0$.

**4.37. Proposition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two* **locally convex** *spaces and $T$ a* **linear** *map from $\mathfrak{X}$ into $\mathfrak{Y}$.* **Assume** *that $\mathfrak{P}$ is a* **basis** *of continuous seminorms on $\mathfrak{X}$ and $\mathfrak{Q}$ is a* **basis** *of continuous seminorms on $\mathfrak{Y}$.* **Then** *the following two assertions are equivalent:*

*i.* *$T: \mathfrak{X} \to \mathfrak{Y}$ is continuous;*

*ii.* *For every continuous seminorm $\mathfrak{q} \in \mathfrak{Q}$, there exists a continuous seminorm $\mathfrak{p} \in \mathfrak{P}$ and a constant $c_{\mathfrak{q}} > 0$ such that*

$$|Tx|_{\mathfrak{q}} \leqslant c_{\mathfrak{q}} |x|_{\mathfrak{p}} \quad \forall x \in \mathfrak{X}. \tag{4.7}$$

*Note that, both $c_{\mathfrak{q}}$ and $\mathfrak{p}$ may depend on $\mathfrak{q}$ but not on $x$.*

**4.38. Remark.** If $\mathfrak{X}, \mathfrak{Y}$ are normed spaces, then $\mathfrak{P} := \{\|\cdot\|_{\mathfrak{X}}\}$ and $\mathfrak{Q} := \{\|\cdot\|_{\mathfrak{Y}}\}$ are bases of continuous seminorms and, therefore, $T$ is continuous if, and only if, there exists a constant $c > 0$ such that

$$\|Tx\|_{\mathfrak{Y}} \leqslant c\, \|x\|_{\mathfrak{X}}$$

for every $x \in \mathfrak{X}$. This is the classical characterization, in normed vector spaces, of linear and continuous operators as bounded operators.

**PROOF.** [*i.* ⇒ *ii.*] Let $T$ be continuous on $\mathfrak{X}$. Due to **Lemma 4.36**, for every $\mathfrak{q} \in \mathfrak{Q}$ there exists a continuous seminorm $\mathfrak{p}_1$ on $\mathfrak{X}$ ($\mathfrak{p}_1$ needs not to be in $\mathfrak{P}$) such that $|Tx|_{\mathfrak{q}} \leqslant |x|_{\mathfrak{p}_1}$. Since $\mathfrak{P}$ is a basis of continuous seminorms, there exists a $\mathfrak{p} \in \mathfrak{P}$ and a $c > 0$ such that $|x|_{\mathfrak{p}_1} \leqslant c|x|_{\mathfrak{p}}$. Overall,

$$|Tx|_{\mathfrak{q}} \leqslant |x|_{\mathfrak{p}_1} \leqslant c|x|_{\mathfrak{p}}.$$

Note that $c$ depends on $\mathfrak{p}_1$ and, therefore, on $\mathfrak{q}$.

[*ii.* ⇒ *i.*] We argue like in **Lemma 4.36**. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a generalized sequence in $\mathfrak{X}$ converging to $0 \in \mathfrak{X}$. The generalized sequence $(|x_\lambda|_{\mathfrak{p}})_{\lambda \in \Lambda}$ converges to $0 \in \mathbb{R}$, and so does (because of (4.7)) the generalized sequence $(|Tx_\lambda|_{\mathfrak{q}})_{\lambda \in \Lambda}$ regardless of the continuous seminorm $q \in \mathfrak{Q}$ we consider. Recalling **Proposition 4.34**, we conclude.

**4.39. Corollary.** *Let $\mathfrak{X}$ be a locally convex space.* **Assume** *that $f: \mathfrak{X} \to \mathbb{K}$ is a linear functional on $\mathfrak{X}$ and $\mathfrak{P}$ a* **basis** *of continuous seminorms on $\mathfrak{X}$.* **Then** *a necessary and sufficient condition for the linear functional $f$ to be* **continuous** *is that there exist $\mathfrak{p} \in \mathfrak{P}$ and $c > 0$ such that*

$$|f(x)| \leqslant c\, |x|_{\mathfrak{p}} \quad \forall x \in \mathfrak{X}.$$

**PROOF.** It is a particular case of **Proposition 4.37** because $\mathbb{K}$ is a locally convex space defined by the basis of continuous seminorms $\mathfrak{Q} = \{|\cdot|\}$ having just one element.

### 4.2.3. Characterization of bounded subsets in locally convex spaces

**4.40. Proposition. Assumptions:** *Let $\mathfrak{X}$ be a locally convex space and $\mathfrak{P}$ a* **basis** *of continuous seminorms on $\mathfrak{X}$.* **Claim:** *A subset $A$ of $\mathfrak{X}$ is bounded* **if, and only if,** *every basic continuous seminorm $\mathfrak{p} \in \mathfrak{P}$ is* **bounded on** *$A$. The expression «$\mathfrak{p}$ is bounded on $A$» means nothing but*

$$\sup_{x \in A} \mathfrak{p}(x) < +\infty.$$

**4.41. Remark.** More explicitly, $A$ is bounded if for every $\mathfrak{p}\in\mathfrak{P}$ there exists a positive constant $\alpha_{\mathfrak{p}}>0$ such that $\mathfrak{p}(x)<\alpha_{\mathfrak{p}}$ for every $x\in A$. In other words, we do not require any uniformity with respect to $\mathfrak{p}\in\mathfrak{P}$. In fact, in general, even if $A$ is bounded, one has $\sup_{\mathfrak{p}\in\mathfrak{P}}\sup_{x\in A}\mathfrak{p}(x)=+\infty$. This can be easily undrstood in the normed space $(\mathbb{R}^2,|\cdot|_2)$ where the family $\mathfrak{p}_\lambda(\cdot):=\lambda|\cdot|_2$ with $\lambda>0$ is a basis of continuous seminorms on $(\mathbb{R}^2,|\cdot|_2)$. Now, if $B$ is the unit ball associated with $|\cdot|_2$, then

$$\sup_{x\in B}\mathfrak{p}_\lambda(x)=\lambda\quad\forall\lambda>0.$$

Therefore even if $B$ is bounded, we have $\sup_{\lambda>0}\sup_{x\in B}\mathfrak{p}_\lambda(x)=+\infty$.

Recall that $A$ absorbe $B$ if there exists a $\lambda_0(B)>0$ ($\lambda_0(B)\neq\infty$) such that $\lambda A\supseteq B$ for every $|\lambda|\geqslant\lambda_0(B)$. In other words, $A$ absorbe $B$ if there exists an $\lambda_0(B)>0$ such that for every $b\in B$ one has $b\in\lambda A$ for every $|\lambda|\geqslant\lambda_0(B)$.

**Proof.** Let $A$ be a bounded subset of $\mathfrak{X}$ and $\mathfrak{p}$ a continuous seminorm on $\mathfrak{X}$. The unit semiball $B_\bullet(\mathfrak{p})$ is a neighborhood of the origin; therefore, the set $A$ is absorbed by $B_\bullet(\mathfrak{p})$. In particular, this means that $A\subseteq\alpha B_\bullet(\mathfrak{p})$ for some $\alpha>0$. Hence

$$\sup_{x\in A}\mathfrak{p}(x)\leqslant\sup_{x\in\alpha B_\bullet(\mathfrak{p})}\mathfrak{p}(x)=\sup_{y\in B_\bullet(\mathfrak{p})}\mathfrak{p}(\alpha y)\leqslant\alpha.$$

**Alternatively**, the first part of the proposition can be proved via **Proposition 3.48** as soon as we note that $\mathfrak{p}$ is, by definition, circled homogeneous as a function from the topological vector space $\mathfrak{X}$ into the topological vector space $\mathbb{R}$.

On the other hand, let us suppose that every continuous seminorm $\mathfrak{p}\in\mathfrak{P}$ is **bounded on** $A$. For any $\mathfrak{p}\in\mathfrak{P}$ there exists $\alpha_{\mathfrak{p}}\in\mathbb{R}_+$ such that

$$\sup_{x\in A}\mathfrak{p}(x)\leqslant\alpha_{\mathfrak{p}}. \tag{4.8}$$

To show that $A$ is bounded, it is sufficient to show (cf. **Proposition 3.42**) that $A$ is absorbed by a fundamental system of neighborhoods of the origin, e.g., by any set of the type $\rho B_\bullet(\mathfrak{p})$ with $\rho>0$ and $\mathfrak{p}\in\mathfrak{P}$. For that, observe that (4.8) implies for every $\mathfrak{p}\in\mathfrak{P}$ and every $\rho>0$, $A\subseteq\alpha_{\mathfrak{p}}B_\bullet(\mathfrak{p})\equiv\left(\frac{\alpha_{\mathfrak{p}}}{\rho}\right)\rho B_\bullet(\mathfrak{p})$. Therefore, $A$ is absorbed by $\rho B_\bullet(\mathfrak{p})$. ■■■■

### 4.2.4. Characterizing continuous bilinear forms in locally convex spaces

**4.42. Proposition. Assumptions:** *Let $\mathfrak{X},\mathfrak{Y}$ and $\mathfrak{Z}$ three locally convex spaces, $\mathfrak{P}_{\mathfrak{X}}$ (resp. $\mathfrak{P}_{\mathfrak{Y}}$, resp. $\mathfrak{P}_{\mathfrak{Z}}$) a basis of continuous seminorms on $\mathfrak{X}$ (resp. $\mathfrak{Y}$, resp. $\mathfrak{Z}$). Let $g$ be a* **bilinear** *map defined in $\mathfrak{X}\times\mathfrak{Y}$ and taking values in $\mathfrak{Z}$.* **Claim:** *The following two assertions are equivalent:*

*i. The bilinear map $g$ is continuous on $\mathfrak{X}\times\mathfrak{Y}$.*

*ii. For every seminorm $\mathfrak{p}_{\mathfrak{Z}}\in\mathfrak{P}_{\mathfrak{Z}}$ there exist a seminorm $\mathfrak{p}_{\mathfrak{X}}\in\mathfrak{P}_{\mathfrak{X}}$, a seminorm $\mathfrak{p}_{\mathfrak{Y}}\in\mathfrak{P}_{\mathfrak{Y}}$, and a constant $c_+\in\mathbb{R}^+$ such that*

$$|g(x,y)|_{\mathfrak{p}_{\mathfrak{Z}}}\leqslant c_+|x|_{\mathfrak{p}_{\mathfrak{X}}}|y|_{\mathfrak{p}_{\mathfrak{Y}}}.$$

**Proof.** Statement ***ii.*** shows that $g$ is continuous at $(0,0)$ and, therefore, continuous everywhere due to **Lemma 3.29**. To show that ***ii.*** implies ***i.***, one can argue as in the proof of **Lemma 4.36**. ■■■■

## 4.3 Fréchet spaces

### 4.3.1. Definition of Fréchet space

**4.43. Definition.** We say that a **locally convex** (topological vector) space $(\mathfrak{X}, \mathcal{V})$ is a **Frechét space** if it has all the following properties:

*i.* The topology $\mathcal{V}$ is Hausdorff separated.

*ii.* The topology $\mathcal{V}$ is first countable, i.e., it admits a countable basis of neighborhoods of the origin.

*iii.* The space $(\mathfrak{X}, \mathcal{V})$ is (Cauchy) complete.

Note that while properties *i.* and *ii.* just depend on the topology of $\mathfrak{X}$, property *iii.* does not make sense in general topological spaces. Indeed, the notion of completeness relies on the notion of Cauchy (generalized) sequence which, the way we defined, depends on the algebraic vector space structure.

The definition of Cauchy (generalized) sequence can be given in the more general contex of uniform spaces, but here we do not dwell on this.

**Reminder [on the notion of complete space].** Let $\mathfrak{X}$ be a topological vector space and $\mathcal{V}(0)$ the filter of neighborhoods of the origin. Let $A$ be a subset of $\mathfrak{X}$ and $(x_\lambda)_{\lambda\in\Lambda}$ a generalized sequence taking values in $A$. We say that the generalized sequence $(x_\lambda)_{\lambda\in\Lambda}$ is **a Cauchy generalized sequence** (or a **Cauchy net**) in $A$, if for every neighborhood $U \in \mathcal{V}(0)$ there exists a $\lambda_0 \in \Lambda$ such that

$$x_{\lambda_1} - x_{\lambda_2} \in U \quad \textbf{whenever} \quad \lambda_1 \succcurlyeq \lambda_0 \textbf{ and } \lambda_2 \succcurlyeq \lambda_0.$$

We say that the set $A \subseteq \mathfrak{X}$ is **complete** (resp. **sequentially complete**) if every Cauchy net (resp. every ordinary Cauchy sequence) on $A$ converges towards an element $a \in A$.

**4.44. Remark.** According to **Proposition 3.38**, condition *iii.* can be replaced by the requirement «*The space* $(\mathfrak{X}, \mathcal{V})$ *is* **sequentially** (Cauchy) *complete*».

### 4.3.2. An equivalent definition of Fréchet space

It is possible to reformulate the definition of Fréchet space by replacing the conditions «*locally convex*» and «*first countable*» with the single requirement that «$\mathfrak{X}$ admits a **countable** basis of **continuous seminorms**». More precisely, the following result holds.

**4.45. Proposition. Assumption:** *Let* $(\mathfrak{X}, \mathcal{V})$ *be a* **topological vector space** *(not necessarily Hausdorff separated or complete).* **Claim:** *The space* $\mathfrak{X}$ *admits a countable basis of continuous seminorms if, and only if, it is locally convex and first countable.*

**4.46. Remark.** We formulated **Definition 4.26** in the context of locally convex spaces. Therefore, officially, one has to clarify what it means for a family of continuous seminorms defined on a (not necessarily a locally convex) topological vector space to be a basis of continuous seminorms. But this is what one can imagine: A *topological vector space* $\mathfrak{X}$ admits a countable basis of continuous seminorms $\{\mathfrak{p}_n\}_{n\in\mathbb{N}}$, if the family $\{\rho B_\bullet(\mathfrak{p}_n)\}_{\rho>0, n\in\mathbb{N}}$ is a fundamental system of neighborhoods of the origin of $\mathfrak{X}$. Although the notion of basis of continuous seminorms makes sense in any topological vector space, **Proposition 4.45** highlights that it is a natural notion in the realm of locally convex spaces.

**Proof.** Let $\mathfrak{X}$ be locally convex and first countable space. Since $\mathfrak{X}$ satisfies the first axiom of countability, it admits a countable basis $(U_n)_{n\in\mathbb{N}}$ of neighborhoods of the origin. Since $\mathfrak{X}$ is locally convex, every $U_n$ contains a barrelled set. Therefore, we can assume that $(U_n)_{n\in\mathbb{N}}$ is a countable basis of barrelled neighborhoods of the origin. Let $\mathfrak{p}_n$ be the gauge of $U_n$. Clearly, $(\mathfrak{p}_n)_{n\in\mathbb{N}}$ is a countable basis of continuous seminorms on $\mathfrak{X}$.

For the other direction, we simply note that if $\mathfrak{X}$ is defined by a countable basis of continuous seminorms $(\mathfrak{p}_n)_{n\in\mathbb{N}}$, then the family

$$\left\{\frac{1}{m}B_\bullet(\mathfrak{p}_n) :: m\in\mathbb{N}^*, n\in\mathbb{N}\right\}$$

is a **countable** basis of convex neighborhoods of the origin of $\mathfrak{X}$ (cf. **Remark 4.7**).

### 4.3.3. Finite-dimensional spaces

**4.47. Definition.** A locally convex space $\mathfrak{X}$ is called **normable** if there exists a norm $\|\cdot\|$ on $\mathfrak{X}$ such that $\{\|\cdot\|\}$ is a basis of continuous seminorms on $\mathfrak{X}$. Every complete normed space is called a Banach space. Every Banach space is a Fréchet space.

Pay attention to the difference bewteen the statements «*every (purely algebraic) and finite dimensional vector space can be structured into a normed space*», and «*every finite dimensional locally convex space is normable*».

We are going to characterize finite-dimensional and (Hausdorff) separated locally convex spaces.

○

**4.48. Lemma.** *Let $X$ be a (purely algebraic) vector space of dimension $n\in\mathbb{N}$. Let $(e_1,\dots,e_n)$ be a basis of $X$. For every $x\in X$ such that $x=\xi_1e_1+\dots+\xi_ne_n$ with $\xi:=(\xi_i)_{i\in\mathbb{N}_n}\in\mathbb{K}^n$, we set*

$$\|x\|_1 := |\xi|_1 \equiv |\xi_1|+\dots+|\xi_n|.$$

*Then, $(X,\|\cdot\|_1)$ is a normed space which is topologically (linearly) isomorphic to the normed space $(\mathbb{K}^n,|\cdot|_1)$. In other words,* **every (purely algebraic) and finite-dimensional vector space can be structured into a normed space**. *Moreover, once endowed $X$ with this norm, it turns out that:*

*i. Every* **seminorm** *$\mathfrak{p}$ on $X$ is* **continuous** *on the normed space $(X,\|\cdot\|_1)$.*

In other words, the construction makes any seminorm continuous on $X$.

*ii. Every* **norm** *$\mathfrak{q}$ on $X$ is* **equivalent** *to the norm $\|\cdot\|_1$.*

*The claim ii. is usually expressed by the motto «***all norms on a finite-dimensional space are equivalent***».*

**4.49. Remark.** Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a locally convex space, are said to be equivalent, if they both are bases of continuous seminorms, i.e., if there exist constant $c_1,c_2>0$ such that

$$c_1\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant c_2\|\cdot\|_1.$$

In other words, the two norms are equivalent if they generate the same (locally convex) topology.

**Proof.** That every (purely algebraic) and finite-dimensional vector space can be structured into a normed space has been implicitly proved in the statement of the lemma. Let us prove claim ***i.*** Let $x=\xi_1e_1+\dots+\xi_ne_n$. We have

$$\mathfrak{p}(x) \leqslant \sum_{i\in\mathbb{N}_n}|\xi_i|\,\mathfrak{p}(e_i) \leqslant \left(\max_{i\in\mathbb{N}_n}\mathfrak{p}(e_i)\right)\|x\|_1.$$

Thus, the seminorm $\mathfrak{p}$ is continuous at the origin and therefore everywhere.

***ii.*** Let $\mathfrak{q}$ be a norm on $X$. Due to claim ***i.***, the norm $\mathfrak{q}$ is continuous on the normed space $(X, \|\cdot\|_1)$. Also, since both $\mathfrak{q}$ and $\|\cdot\|_1$ are norms on $X$, they vanish only at $x=0$. Hence the «ratio» function $r: x \mapsto \mathfrak{q}(x) / \|x\|_1$ is well-defined and continuous on $X \setminus \{0\}$. The equivalence of $\mathfrak{q}$ and $\|\cdot\|_1$ amounts to prove that $r$ is bounded in $X \setminus \{0\}$ and lower bounded by a strictly positive constant. For that, observe that $r$ is well-defined and continuous on the **normed unit sphere** $\mathbb{S}_1^{n-1} = \{x \in X :: \|x\|_1 = 1\}$. But $\mathbb{S}_1^{n-1}$ is a compact set, because $(X, \|\cdot\|_1)$ is topologically isomorphic to $(\mathbb{K}^n, |\cdot|_1)$. Therefore, $r$ is bounded on $\mathbb{S}_1^{n-1}$ and by homogeneity

$$\mathfrak{q}(x) \leqslant \left( \max_{\sigma \in \mathbb{S}_1^{n-1}} \mathfrak{q}(\sigma) \right) \|x\|_1. \tag{4.9}$$

on $X \setminus \{0\}$. Similarly, again by homogeneity,

$$\mathfrak{q}(x) \geqslant \left( \min_{\sigma \in \mathbb{S}_1^{n-1}} \mathfrak{q}(\sigma) \right) \|x\|_1. \tag{4.10}$$

This concludes the proof. ■■■■

**4.50. Remark.** Since $\mathfrak{X} := (X, \|\cdot\|_1)$ is a normed space, $\|\cdot\|_1$ is a basis of continuous seminorms on $\mathfrak{X}$. According to **Proposition 4.28**, for every continuous seminorm $\pi$ on $\mathfrak{X}$, there exists a constant $c_\pi > 0$ such that $\pi(x) \leqslant c_\pi \|x\|_1$ for every $x \in \mathfrak{X}$. In particular, since $\mathfrak{q}$ is continuous on $\mathfrak{X}$, we have

$$\mathfrak{q}(x) \leqslant c_{\mathfrak{p}} \|x\|_1$$

for evey $x \in X$. Therefore, the proof of **Lemma 4.48** is needed to derive the strictly positive lower bound on $r: x \mapsto \mathfrak{q}(x) / \|x\|_1$. However, the constants derived in the expressions (4.9) and (4.10) during the proof of **Lemma 4.48** are sharp.

So far, we have proved that every finite-dimensional (and purely) algebraic vector space can be structured into a normed vector space. We achieved this by transferring the topological structure of the normed space $(\mathbb{K}^n, |\cdot|_1)$ to the vector space $X$, via the classical (and non-natural) linear isomorphism which, by the way, depends on the choice of a basis of $X$. Also, we have shown that any other norm on $X$ would have generated the same topology on $X$. However, so far, we have not proved that every finite-dimensional and **locally convex space** is *normable*. This is the topic of the next result.

Is $\mathbb{K}^n$ normable? Well, the answer does not make sense until one specifies which topology we are considering on it. Does the vector space $\mathbb{K}^n$ admits a norm. Yes, for example $\|\cdot\|_1$.

**4.51. Theorem.** *Every locally convex and* **Hausdorff separated** *space* $\mathfrak{X}$*, having finite dimension* $n \in \mathbb{N}$*, is* **normable** *and, therefore, topologically isomorphic to* $\mathbb{K}^n$*.*

**4.52. Remark.** Note that the notion of «*normable* space» makes sense only in locally convex spaces, as every normed space is locally convex.

**PROOF.** It is sufficient to show that there is a norm $\mathfrak{q}$ on $\mathfrak{X}$ **which is continuous** — the result follows from **Proposition 4.28** and the previous **Lemma 4.48**.

○

We argue by induction. Denote by $(e_1, \ldots, e_n)$ a basis of $\mathfrak{X}$. Also, denote by $\mathfrak{X}_k$ the $k$-dimensional subspace spanned by $(e_1, \ldots, e_k)$ $(1 \leqslant k \leqslant n)$, and by $\mathfrak{B}_k$ the one-dimensional subspace spanned by $(e_k)$.

**($k=1$)** Since $\mathfrak{X}$ is Hausdorff separated, there exists a **continuous seminorm** $\mathfrak{p}_1$ on the whole of $\mathfrak{X}$ whose restriction to $\mathfrak{X}_1$ is a **norm** on $\mathfrak{X}_1$ (cf. Remark 4.10). Let $\mathfrak{K}_1$ be the vector subspace of $\mathfrak{X}_2$ on which $\mathfrak{p}_1$ is zero, that is, the kernel of the restriction of $\mathfrak{p}_1$ to $\mathfrak{X}_2$. Clearly, $\dim \mathfrak{K}_1 \leqslant 1$ because $\mathfrak{p}_1$ is a norm on $\mathfrak{X}_1$. Since the topology of $\mathfrak{X}$ is Hausdorff separated, there exists a **continuous seminorm** $\pi_1$ defined on the whole $\mathfrak{X}$ such that its restriction to $\mathfrak{K}_1$ is a **norm**.

$$\pi_1|\mathfrak{K}_1 \text{ *norm* on } \mathfrak{K}_1 := \ker(\mathfrak{p}_1|\mathfrak{X}_2) \subset \mathfrak{X}_2, \qquad \mathfrak{p}_1|\mathfrak{X}_1 \text{ *norm* on } \mathfrak{X}_1$$

$$\mathfrak{X}_1 \trianglelefteq \mathfrak{X}_2 \trianglelefteq \cdots \trianglelefteq \mathfrak{X}_n := \mathfrak{X}$$

As $\mathfrak{X}_2 = \mathfrak{X}_1 \oplus \mathfrak{B}_2$, setting $\mathfrak{p}_2 := \pi_1 \vee \mathfrak{p}_1$, we obtain a *continuous* seminorm on the whole of $\mathfrak{X}$, whose restriction to $\mathfrak{X}_2$ is a norm. Indeed, for any $x = x_1 \oplus b_2 \in \mathfrak{X}_2$

$$\begin{aligned} \mathfrak{p}_2(x_1 \oplus b_2) = 0 &\Leftrightarrow \pi_1(x_1 \oplus b_2) = 0 \quad \textbf{and} \quad \mathfrak{p}_1(x_1 \oplus b_2) = 0 \\ &\Leftrightarrow \pi_1(x_1 \oplus b_2) = 0 \quad \textbf{and} \quad x_1 \oplus b_2 \in \mathfrak{K}_1. \end{aligned}$$

Since $\pi_1$ is a *norm* on $\mathfrak{K}_1$ we conclude that $x_1 \oplus b_2 = 0$. Hence, $\mathfrak{p}_2 := \pi_1 \vee \mathfrak{p}_1$ is a norm on $\mathfrak{X}_2$.

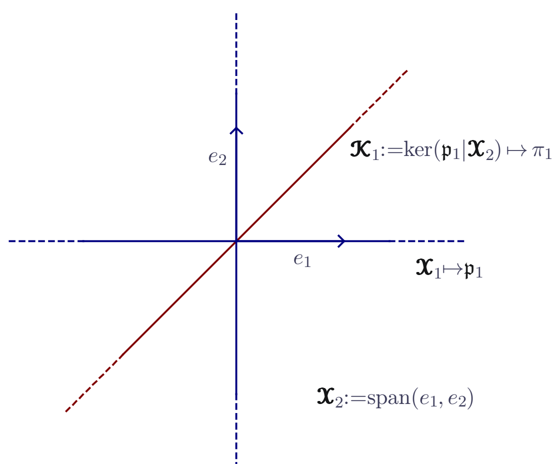



**Figure 4.1.** As $\mathfrak{X}_2 = \mathfrak{X}_1 \oplus \mathfrak{B}_2$, setting $\mathfrak{p}_2 := \pi_1 \vee \mathfrak{p}_1$, we obtain a *continuous* seminorm on the whole of $\mathfrak{X}$, whose restriction to $\mathfrak{X}_2$ is a norm.

**($k$ general)** Assume the existence of a **continuous seminorm** $\mathfrak{p}_k$, defined on the whole $\mathfrak{X}$ and whose restriction to $\mathfrak{X}_k$ is a **norm** on $\mathfrak{X}_k$, and let us show that there exists a seminorm $\mathfrak{p}_{k+1}$ on $\mathfrak{X}$ whose restriction to $\mathfrak{X}_{k+1}$ is a norm. Let $\mathfrak{K}_k$ be the vector subspace of $\mathfrak{X}_{k+1}$ on which $\mathfrak{p}_k$ is zero, that is, the kernel of the restriction of $\mathfrak{p}_k$ to $\mathfrak{X}_{k+1}$. Clearly, $\dim \mathfrak{K}_k \leqslant 1$ because $\mathfrak{p}_k$ is a norm on $\mathfrak{X}_k$. Since the topology of $\mathfrak{X}$ is Hausdorff separated, there exists a **continuous seminorm** $\pi_k$ on $\mathfrak{X}$ whose restriction to $\mathfrak{K}_k$ is a **norm**.

$$\pi_k|\mathfrak{K}_k \text{ *norm* on } \mathfrak{K}_k := \ker(\mathfrak{p}_k|\mathfrak{X}_{k+1}) \subset \mathfrak{X}_{k+1}, \qquad \mathfrak{p}_k|\mathfrak{X}_k \text{ *norm* on } \mathfrak{X}_k$$

$$\mathfrak{X}_1 \trianglelefteq \cdots \trianglelefteq \mathfrak{X}_k \trianglelefteq \mathfrak{X}_{k+1} \trianglelefteq \cdots \trianglelefteq \mathfrak{X}_n := \mathfrak{X}$$

Indeed, as $\dim \mathfrak{K}_k \leqslant 1$, there exists $u_k \in \mathfrak{X}$ such that $\mathfrak{K}_k = \{\lambda u_k\}_{\lambda \in \mathbb{K}}$. If $u_k = 0$, then the null functional answers to the question. Otherwise, since $\mathfrak{X}$ is Hausdorff separated, there exists a continuous seminorm $\pi_k$ on $\mathfrak{X}$ such that $\pi_k(u_k) \neq 0$, and by homogeneity $\pi_k(\lambda u_k) = 0$ if, and only if $\lambda = 0$.

We set $\mathfrak{p}_{k+1} := \pi_k \vee \mathfrak{p}_k$. Then $\mathfrak{p}_{k+1}$ is a **continuous seminorm** on $\mathcal{X}$ whose restriction to $\mathcal{X}_{k+1}$ is a **norm** on $\mathcal{X}_{k+1}$. Indeed, as $\mathcal{X}_{k+1} = \mathcal{X}_k \oplus \mathcal{B}_{k+1}$, setting $\mathfrak{p}_{k+1} := \pi_k \vee \mathfrak{p}_k$, we obtain a seminorm on the whole of $\mathcal{X}$ whose restriction to $\mathcal{X}_{k+1}$ is a norm. Indeed, for any $x = x_k \oplus b_{k+1} \in \mathcal{X}_{k+1}$

$$\begin{aligned} \mathfrak{p}_{k+1}(x_k \oplus b_{k+1}) = 0 &\Leftrightarrow \pi_k(x_k \oplus b_{k+1}) = 0 \quad \textbf{and} \quad \mathfrak{p}_k(x_k \oplus b_{k+1}) = 0 \\ &\Leftrightarrow \pi_k(x_k \oplus b_{k+1}) = 0 \quad \textbf{and} \quad x_k \oplus b_{k+1} \in \mathcal{K}_k. \end{aligned}$$

But $\pi_k$ is a *norm* on $\mathcal{K}_k$ and therefore $x_k \oplus b_{k+1} = 0$. Hence, $\mathfrak{p}_{k+1} := \pi_k \vee \mathfrak{p}_k$ is a norm on $\mathcal{X}_{k+1}$.

Eventually, we set $\mathfrak{q} := \mathfrak{p}_n$ and, in this way, we get that $\mathfrak{q}$ is a continuous norm on $\mathcal{X}$. ■

## 4.4 An alternative definition of Fréchet space

Here we want to give an alternative definition of Fréchet space, based on the concept of metrizability.

**4.53. Definition.** A topological vector space $(\mathcal{X}, \tau)$ is **metrizable** if there exists a metric $d: \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ such that $\tau$ coincides with the **metric topology** defined by the distance $d$.

We need to recall the following characterization of metrizable topological vector spaces, whose proof can be found, e.g., in [**Theorem 5.10**, **p. 172**, ALIPRANTIS, D., BORDER, K. *Infinite Dimensional Analysis: a Hitchhiker's guide*. Springer, Heidelberg, **2006**].

**4.54. Theorem. (Birkhoff 1936[4.1], Kakutani 1936[4.2]) Metrizability theorem:** *A topological vector space $(\mathcal{X}, \tau)$ is* **metrizable** *if, and only if, it is Hausdorff separated and satisfies the first axiom of countability (that is, it admits a countable filter base of neighborhoods of the origin). Moreover, if $\mathcal{X}$ is metrizable, the metric $d: \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, inducing $\tau$, can be chosen to be* **translation invariant**, *that is*

$$d(x+z, y+z) = d(x, y) \quad \forall x, y \in \mathcal{X}.$$

We can therefore replace **Definition 4.43** of Fréchet space with the following equivalent one.

**4.55. Definition.** We call **Fréchet space** any locally convex space which is *metrizable* and *complete*. We call **pre-Fréchet space** any locally convex space which is metrizable.

The notion of pre-Fréchet space is borrowed from the common practice to use the term **pre-Hilbert** spaces to refer to spaces that become Hilbert spaces after completion (i.e., inner product spaces). In other words, a Fréchet space is a complete pre-Fréchet space.

Then, from **Proposition 4.45** and **Theorem 4.54**, we get the following result.

**4.56. Theorem.** *A topological vector space is a pre-Fréchet space if, and only if, it is Hausdorff separated and admits a countable basis of continuous seminorms.*

**PROOF.** Indeed, according to **Proposition 4.45**, a topological vector space (not necessarily Hausdorff separated or complete) admits a countable basis of continuous seminorms if, and only if, it is a first countable locally convex space. But then, the metrizability theorem (**Theorem 4.54**) completes the proof. ■

**Metrization theorems.** A metrization theorem is a result that gives sufficient conditions, and sometimes necessary and sufficient conditions, for a topological space to be metrizable, i.e., for its topology to be induced by a metric.

An optimal metrization theorem, giving a necessary and sufficient condition, is Nagata-Smirnov *metrization theorem*:

**4.57. Theorem. (Nagata-Smirnov)** *A topological space is metrizable* **if**, **and only if**, *it is regular, Hausdorff, and has a* countably locally finite base.

A variation of this, directly implied by the fact that metrizable spaces have countably locally discrete bases, is the Bing *metrization theorem*:

**4.58. Theorem. (Bing)** *A topological space is metrizable if, and only if, it is regular, Hausdorff, and has a countably locally discrete topological base.*

A historical predecessor and direct implication of these theorems is the **Urysohn** metrization theorem:

**4.59. Theorem. (Urysohn)** *Every second-countable, regular, Hausdorff space is metrizable.*

**4.60. Remark.** These results assume Hausdorff separation and regularity as essential properties (valid in any metric space) and differ on a structural condition related to a topological basis or a local basis. Recalling that every topological vector space is regular and that second-countable spaces are a subset of first-countable ones, it appears clear that the interesting part in Birkhoff–Kakutani result is in that it gives a *necessary* and *sufficien*t condition under the hypothesis that the topological vector space is *first countable*. This, of course, is possible because we are treating topological vector spaces and not abstract topological spaces.

# 5

# Inductive Topologies

## 5.1 | The induced vector subspace topology

We introduced the concept of subspace topology in **Definition 2.13** when we were in the general topology framework. Here we revisit the notion in the context of locally convex spaces.

Let $\mathfrak{X}$ be a topological vector space, $\mathcal{V}(0)$ the filter of neighborhoods of the origin of $\mathfrak{X}$ and $\mathfrak{M} \trianglelefteq \mathfrak{X}$ a (purely algebraic) vector subspace of $\mathfrak{X}$. We set

$$\mathcal{V}_{\mathfrak{M}}(0) := \{V_{\mathfrak{M}} \subseteq \mathfrak{M} :: V_{\mathfrak{M}} = V \cap \mathfrak{M} \text{ for some } V \in \mathcal{V}(0)\}. \tag{5.1}$$

It is easily seen that $\mathcal{V}_{\mathfrak{M}}(0)$ satisfies the properties of the **structure theorem** (cf. **Theorem 3.17** and **Proposition 3.25**) and, therefore, it is a filter of neighborhoods of $0 \in \mathfrak{M}$ for a topology compatible with the vector space structure of $\mathfrak{M}$. We say that the topology generated by the filter (5.1), is **induced** on $\mathfrak{M}$ by $\mathfrak{X}$.

**Notation 5.1.** *Sometimes we write $\mathcal{V} \mid \mathfrak{M}$ to denote the topology induced on $\mathfrak{M}$ by $\mathfrak{X}$.*

**5.2. Remark.** By **Remark 2.14** on the trace of a filter, we can say that the topology induced by $(\mathfrak{X}, \mathcal{V}_{\mathfrak{X}})$ on a subspace $\mathfrak{M} \trianglelefteq \mathfrak{X}$ is the function which sends each $m \in \mathfrak{M}$ to the trace of $\mathcal{V}_{\mathfrak{X}}(m)$ on $\mathfrak{M}$.

**5.3. Proposition.** *If $\mathfrak{X}$ is a locally convex space and $\mathfrak{M} \trianglelefteq \mathfrak{X}$, then $\mathfrak{M}$ is also a locally convex space (when endowed with the topology induced on it by $\mathfrak{X}$).*

**Proof.** It is a consequence of the fact that the intersection of convex sets is still convex, together with the following simple observation: if $\mathcal{B}(0)$ is *any* filter base of neighborhoods of the origin for $\mathfrak{X}$, then the family

$$\mathcal{B}_{\mathfrak{M}}(0) := \{B_{\mathfrak{M}} \subseteq \mathfrak{M} :: B_{\mathfrak{M}} = B \cap \mathfrak{M} \text{ for some } B \in \mathcal{B}(0)\},$$

i.e., the trace of $\mathcal{B}(0)$ on $\mathfrak{M}$, is a filter base of neighborhoods of the origin for $\mathfrak{M}$ (endowed with the topology induced by $\mathfrak{X}$).

But now, if $\mathcal{B}(0)$ is a filter basis of **convex** neighborhoods of the origin of $\mathfrak{X}$ (whose existence is guaranteed by the local convexity assumption on $\mathfrak{X}$) then $\mathcal{B}_{\mathfrak{M}}(0)$, which is a filter base of neighborhoods of the origin for $\mathfrak{M}$, consists of sets of the form $B \cap \mathfrak{M}$ with both $B$ and $\mathfrak{M}$ convex. Hence, $\mathfrak{M}$ is a locally convex space. ■

What we really proved is that a locally convex topology induces a locally convex subspace topology on any of its convex subsets. However, to talk about locally convex spaces we must deal with vector subspaces rather than convex subsets.

The next two results will play a crucial role in proving the the Dieudonné–Schwartz (**Theorem 5.19**).

**5.4. Lemma.** **Assumptions:** *Let* $\boldsymbol{\mathfrak{X}}$ *be a* **locally convex** (*topological vector*) **space**, *and* $\boldsymbol{\mathfrak{M}} \trianglelefteq \boldsymbol{\mathfrak{X}}$ *a vector subspace of* $\boldsymbol{\mathfrak{X}}$. **Claim:** *Given any* $U \in \mathcal{V}_{\boldsymbol{\mathfrak{M}}}(0)$, **if** $U$ *is* **convex**, *then there exists a* convex *neighborhood* $V_c \in \mathcal{V}(0)$ *such that* $U = V_c \cap \boldsymbol{\mathfrak{M}}$.

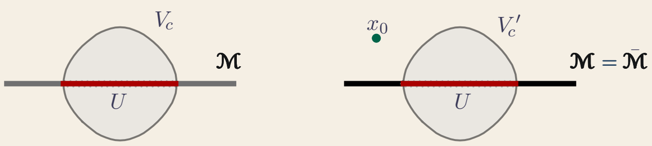



**Figure 5.1.** **(left)** Given any $U \in \mathcal{V}_{\boldsymbol{\mathfrak{M}}}(0)$, if $U$ is convex, then there exists a convex set $V_c \in \mathcal{V}(0)$ such that $U = V_c \cap \boldsymbol{\mathfrak{M}}$. **(right)** If $\boldsymbol{\mathfrak{M}}$ is closed, given any $U \in \mathcal{V}_{\boldsymbol{\mathfrak{M}}}(0)$ with $U$ convex, and any $x_0 \notin \boldsymbol{\mathfrak{M}}$, there exists a convex set $V_c' \in \mathcal{V}(0)$ such that $U = V_c' \cap \boldsymbol{\mathfrak{M}}$ and $x_0 \notin V_c'$.

**5.5. Remark.** We already know that in *any* topological vector space $\boldsymbol{\mathfrak{X}}$, for every $U \in \mathcal{V}_{\boldsymbol{\mathfrak{M}}}(0)$ there exists a $V_{(U)} \in \mathcal{V}(0)$, not necessarily convex, such that $U = V_{(U)} \cap \boldsymbol{\mathfrak{M}}$. **Lemma 5.4** gives more information. If $\boldsymbol{\mathfrak{X}}$ is a **locally convex** space and $U \in \mathcal{V}_{\boldsymbol{\mathfrak{M}}}(0)$ is **convex**, then the neighborhood $V_{(U)} \in \mathcal{V}(0)$ can be chosen to be convex.

**5.6. Lemma.** **Assumptions:** *Let* $\boldsymbol{\mathfrak{X}}$ *be a* **locally convex** (*topological vector*) **space**, *and* $\boldsymbol{\mathfrak{M}} \trianglelefteq \boldsymbol{\mathfrak{X}}$ *a* **closed** *subspace of* $\boldsymbol{\mathfrak{X}}$. **Claim:** *Given any* $U \in \mathcal{V}_{\boldsymbol{\mathfrak{M}}}(0)$ *with* $U$ *convex, and any* $x_0 \notin \boldsymbol{\mathfrak{M}}$, *there exists a convex set* $V_c' \in \mathcal{V}(0)$ *such that* $U = V_c' \cap \boldsymbol{\mathfrak{M}}$ *and* $x_0 \notin V_c'$.

**5.7. Remark.** In any topological vector space $\boldsymbol{\mathfrak{X}}$, for every $U \in \mathcal{V}_{\boldsymbol{\mathfrak{M}}}(0)$ there exists a $V_{(U)} \in \mathcal{V}(0)$, not necessarily convex, such that $U = V_{(U)} \cap \boldsymbol{\mathfrak{M}}$. Here, **Lemma 5.6** assures that if $U$ is **convex** and $\boldsymbol{\mathfrak{M}}$ is **closed** in $\boldsymbol{\mathfrak{X}}$, then for any $x_0 \notin \boldsymbol{\mathfrak{M}}$ the neighborhoods $V_{(U)} \in \mathcal{V}(0)$ can be chosen to be convex and such that $x_0 \notin V_{(U)}$.

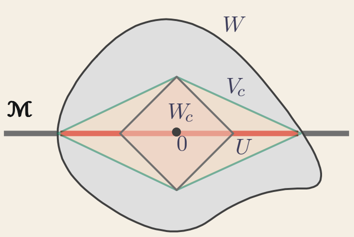



**Figure 5.2.** The geometric construction used in the proof of **Lemma 5.4**. Since $\boldsymbol{\mathfrak{X}}$ is locally convex, the set $W$ contains a convex set $W_c$ which still belongs to $\mathcal{V}(0)$. Here $V_c := K(U \cup W_c)$ is the convex hull of $U \cup W_c$.

**PROOF.** **(of Lemma 5.4)** By definition, $U \in \mathcal{V}_{\boldsymbol{\mathfrak{M}}}(0)$ if, and only if, there exists $W \in \mathcal{V}(0)$ such that $U := W \cap \boldsymbol{\mathfrak{M}}$. Since $\boldsymbol{\mathfrak{X}}$ is locally convex, the set $W$ contains a convex set $W_c$ which still belongs to $\mathcal{V}(0)$. Let us denote by $V_c := K(U \cup W_c)$ the convex hull of $U \cup W_c$. We are going to show that

$$U (:= W \cap \boldsymbol{\mathfrak{M}}) \equiv V_c \cap \boldsymbol{\mathfrak{M}}, \tag{5.2}$$

and this will complete the proof because $V_c$ is convex and in $\mathcal{V}(0)$ (as $V_c \supseteq W_c$ with $W_c \in \mathcal{V}(0)$).

For that, first we observe that since $V_c$ is the convex hull of the *union of two convex sets*, as it is easy to prove, $V_c$ coincides with the union of the family

$U$ convex by hypothesis while $W_c$ is convex by construction

$$(\lambda U + \mu W_c)_{(\lambda,\mu) \in \mathbb{R}_+^2} \quad \text{with } \lambda + \mu = 1.$$

In other words, any $x \in V_c$ is a convex combination of an element $u \in U$ and an element $w \in W_c$.

Clearly, one has $U \subseteq V_c \cap \boldsymbol{\mathcal{M}}$ because, by assumption $U \subseteq \boldsymbol{\mathcal{M}}$ and, on the other hand, $U \subseteq K(U \cup W_c) = V_c$. To show that $V_c \cap \boldsymbol{\mathcal{M}} \subseteq U (:= W \cap \boldsymbol{\mathcal{M}})$, we consider a generic element

$$x = \lambda u + \mu w \in V_c \cap \boldsymbol{\mathcal{M}}, \tag{5.3}$$

with $(\lambda, \mu) \in \mathbb{R}^2_+$, $\lambda + \mu = 1$, $u \in U$, $w \in W_c$, and we show that $x \in U$.

If $\mu = 0$, then $x = u \in U$. On the other hand, since $x, u$ are both elements of $\boldsymbol{\mathcal{M}}$, we have that $w \in \boldsymbol{\mathcal{M}}$ as soon as $\mu \neq 0$. Therefore, if $\mu \neq 0$, we have $w \in W_c \cap \boldsymbol{\mathcal{M}} \subseteq U (:= W \cap \boldsymbol{\mathcal{M}})$, and this concludes the proof because $x$ can be written as a convex combination of two elements of the convex set $U$.

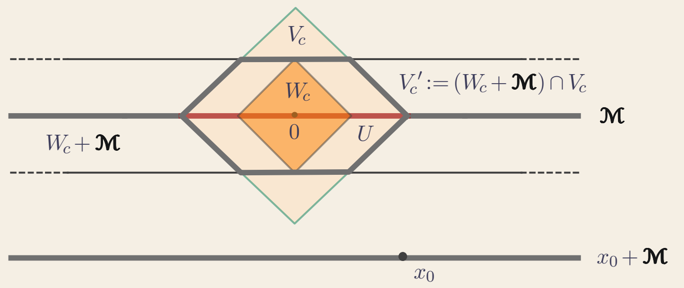



**Figure 5.3.** The geometric construction used in the proof of Lemma 5.6.

**Proof. (of Lemma 5.6)** According to Lemma 5.4, given $U \in \mathcal{V}_{\boldsymbol{\mathcal{M}}}(0)$, with $U$ convex, there exists a convex set $V_c \in \mathcal{V}(0)$ such that $U = V_c \cap \boldsymbol{\mathcal{M}}$.

Now, we observe that the set $x_0 + \boldsymbol{\mathcal{M}}$ is closed and that $0 \notin x_0 + \boldsymbol{\mathcal{M}}$ because $x_0 \notin \boldsymbol{\mathcal{M}}$. Indeed, recall that for $x_1, x_2 \in \boldsymbol{\mathcal{X}}$ there holds $x_1 + \boldsymbol{\mathcal{M}} = x_2 + \boldsymbol{\mathcal{M}}$ if, and only if, $x_2 - x_1 \in \boldsymbol{\mathcal{M}}$; and if $x_2 - x_1 \notin \boldsymbol{\mathcal{M}}$ then $(x_1 + \boldsymbol{\mathcal{M}}) \cap (x_2 + \boldsymbol{\mathcal{M}}) = \emptyset$. This implies that the open set $(x_0 + \boldsymbol{\mathcal{M}})^{\complement}$ is a neighborhood of zero in $\boldsymbol{\mathcal{X}}$; thus, there exists a convex neighborhood $W_c \in \mathcal{V}(0)$ of the origin (in $\boldsymbol{\mathcal{X}}$), that we can always suppose included in $V_c$, such that

$$(x_0 + \boldsymbol{\mathcal{M}}) \cap W_c = \emptyset \quad (\text{and } W_c \subseteq V_c).$$

But the previous relation implies that $w \notin x_0 + \boldsymbol{\mathcal{M}}$ for every $w \in W_c$, i.e., $w - x_0 \notin \boldsymbol{\mathcal{M}}$ for every $w \in W_c$. Hence, $(w + \boldsymbol{\mathcal{M}}) \cap (x_0 + \boldsymbol{\mathcal{M}}) = \emptyset$ for every $w \in W_c$, from which we infer that

$$(W_c + \boldsymbol{\mathcal{M}}) \cap (x_0 + \boldsymbol{\mathcal{M}}) = \emptyset.$$

We then set

Recall that the sum of convex sets is still a convex set

$$V_c' := (W_c + \boldsymbol{\mathcal{M}}) \cap V_c$$

and we conclude the proof as soon as we show that $x_0 \notin V_c'$, $V_c'$ is convex, $V_c' \in \mathcal{V}(0)$, and

$$U = V_c' \cap \boldsymbol{\mathcal{M}}.$$

For that, observe that $x_0 \notin V_c'$ because $x_0 \in (x_0 + \boldsymbol{\mathcal{M}})$ and $x_0 + \boldsymbol{\mathcal{M}}$ is disjoint from $W_c + \boldsymbol{\mathcal{M}}$ and therefore also from its subset $V_c'$. Also, the set $V_c'$ is convex because it is the intersection of the convex set $V_c$ with $W_c + \boldsymbol{\mathcal{M}}$ which is convex because it is the Minkowski sum of two convex sets. Moreover, $V_c' \in \mathcal{V}(0)$, because it is the intersection of two neighborhoods of the origin (note that $(W_c + \boldsymbol{\mathcal{M}}) \supseteq W_c$). Eventually, $U = V_c' \cap \boldsymbol{\mathcal{M}}$ because of

In general, for subsets $A, B \subseteq X$ one has $K(A+B) = K(A) + K(B)$ where $K$ is the convex hull operator.

$$U = \boldsymbol{\mathcal{M}} \cap (V_c \cap \boldsymbol{\mathcal{M}}) \subseteq \underbrace{(W_c + \boldsymbol{\mathcal{M}}) \cap (V_c \cap \boldsymbol{\mathcal{M}})}_{= V_c' \cap \boldsymbol{\mathcal{M}}} \subseteq V_c \cap \boldsymbol{\mathcal{M}} = U.$$

This concludes the proof.

Combining **Lemma 5.4** and **Lemma 5.6** we can infer a refined version of **Lemma 5.6**. Compared to **Lemma 5.6**, here the point $x_0$ that in **Lemma 5.6** is assumed to be outside of $\mathcal{M}$ is now assumed only to be outside of a convex neighborhood in $\mathcal{V}_{\mathcal{M}}(0)$.

**5.8. Corollary. Assumptions:** *Let* $\mathcal{X}$ *be a* **locally convex** (*topological vector*) **space**, *and* $\mathcal{M} \trianglelefteq \mathcal{X}$ *a* **closed** *subspace of* $\mathcal{X}$. **Claim:** *Given any* $U \in \mathcal{V}_{\mathcal{M}}(0)$ *with* $U$ *convex, and any* $x_0 \notin U$, *there exists a convex set* $W \in \mathcal{V}(0)$ *such that* $U = W \cap \mathcal{M}$ *and* $x_0 \notin W$.

PROOF. It is an immediate consequence of **Lemma 5.4** and **Lemma 5.6**. Indeed, by **Lemma 5.4** there exists a *convex* neighborhood $V \in \mathcal{V}(0)$ such that $U = V \cap \mathcal{M}$. Now, if $x_0 \notin V$ we set $W := V$. Otherwise, if $x_0 \in V$ then necessarily $x_0 \notin \mathcal{M}$, and by **Lemma 5.6** there exists $W \in \mathcal{V}(0)$ such that $U = W \cap \mathcal{M}$ and $x_0 \notin W$. ■

## 5.2 Strict inductive limit of locally convex spaces

We start by setting up the framework and by describing the goal of this section.

**Assumptions:** Let $X$ be a (purely algebraic) vector space over $\mathbb{K}$ (as usual, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), and let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ be an *increasing* sequence of vector *sub*spaces of $X$ that *covers* $X$:

$$\mathcal{X}_n \trianglelefteq \mathcal{X}_{n+1} \quad \textbf{and} \quad \bigcup_{n\in\mathbb{N}} \mathcal{X}_n = X. \tag{5.4}$$

Assume that:

*i.* Every subspace $\mathcal{X}_n$ is endowed with a locally convex topology $\mathcal{V}_n$ compatible with the vector space structure. In other words, we are considering a sequence $(\mathcal{X}_n, \mathcal{V}_n)_{n\in\mathbb{N}}$ of *locally convex spaces*.

*ii.* The topology $\mathcal{V}_n$ coincides with the topology induced by $\mathcal{V}_{n+1}$ on $\mathcal{X}_n$ ($\trianglelefteq \mathcal{X}_{n+1}$). In other words $\mathcal{V}_n = \mathcal{V}_{n+1}|\mathcal{X}_n$, that is

$$\mathcal{V}_n(0) := \{V_n \subseteq \mathcal{X}_n :: V_n = V_{n+1} \cap \mathcal{X}_n \textbf{ for some } V_{n+1} \in \mathcal{V}_{n+1}(0)\}.$$

Here, we have denoted by $\mathcal{V}_n(0) := \mathcal{V}_{\mathcal{X}_n}(0)$ the neighborhood topology induced by the locally convex space $(\mathcal{X}_{n+1}, \mathcal{V}_{n+1})$ on $\mathcal{X}_n$. Note that, by **Lemma 5.4**, every *convex* neighborhood $U_n \in \mathcal{V}_n(0)$ can be realized as $U_n = U_{n+1} \cap \mathcal{X}_n$ for some *convex* neighborhood $U_{n+1} \in \mathcal{V}_{n+1}(0)$.

**Aim:** Until now, it has not been defined any topology on $X$. We aim to endow the vector space $X$ with a locally convex topology *compatible* (in a sense to be made precise soon) with the sequence of locally convex spaces $(\mathcal{X}_n, \mathcal{V}_n)_{n\in\mathbb{N}}$.

To this end, we denote by $\mathcal{B}_\diamond(X)$ the family of all subsets of $X$, which are absorbing, balanced, and convex. Note that the family $\mathcal{B}_\diamond(X)$ consists of sets defined in a purely algebraic way.

**5.9. Proposition. Assumptions:** *Let* $\mathcal{B} \subseteq \mathcal{B}_\diamond(X)$ *be the family of those elements* $B$ *in* $\mathcal{B}_\diamond(X)$ *such that, for every* $n \in \mathbb{N}$, $B \cap \mathcal{X}_n$ *is a neighborhood of the origin in* $\mathcal{X}_n$. *In other words, define*

$$\big(B \in \mathcal{B}\big) \quad \textbf{if, and only if,} \quad \big(B \in \mathcal{B}_\diamond(X) \textbf{ and } \forall n \in \mathbb{N}\,[B \cap \mathcal{X}_n \in \mathcal{V}_n(0)]\big).$$

*Note that* $\mathcal{B}$ *is nonempty because the whole space* $X$ *always belongs to* $\mathcal{B}$.

**Claim:** $\mathcal{B}$ *is a* **filter base** *of neighborhoods of the origin of* $X$ *for a locally convex topology* $\mathcal{V}$ *which is compatible with the vector space structure. In other words,* $\mathcal{B}$ *induces a neighborhood topology* $\mathcal{V}$ *on* $X$ *that turns* $(X, \mathcal{V})$ *into a* **locally convex space**.

We then say that the **locally convex space** $(X, \mathcal{V})$, is **the strict inductive limit** of the sequence of locally convex spaces $(\mathfrak{X}_n, \mathcal{V}_n)_{n\in\mathbb{N}}$. The sequence $(\mathfrak{X}_n, \mathcal{V}_n)_{n\in\mathbb{N}}$ is called **a defining sequence for** $(X, \mathcal{V})$. Sometimes, the notation

$$(\mathfrak{X}, \mathcal{V}) := \lim (\mathfrak{X}_n, \mathcal{V}_n) \tag{5.5}$$

is used to denote this circumstance — the notation is borrowed from category theory, where the inductive limit is also known as the *direct* limit. With this definition in mind, we can restate **Proposition 5.9** as follows.

**5.10. Proposition.** *The inductive limit* $(\mathfrak{X}, \mathcal{V}) := \lim (\mathfrak{X}_n, \mathcal{V}_n)$ *of the* increasing *sequence of locally convex spaces* $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ *is a locally convex space.*

**Note 5.11.** Before giving the proof, let us recall the statement of **Proposition 4.5**, which will be used in the proof. **Assumptions:** Let $X$ be a (purely algebraic) vector space and $\mathcal{S}$ a **filter base** on $X$ consisting of sets that are at the same time absorbing, balanced, and convex. **Claim:** The family $\mathcal{B} := \cup_{\lambda\in\mathbb{R}_+^*} \lambda\mathcal{S}$, consisting of the sets obtained by the elements of $\mathcal{S}$ via any homothétic transformation of strictly positive ratio, is still a filter base and, actually, a fundamental system of neighborhoods of the origin for a locally convex topology on $X$ compatible with the vector space structure of $X$.

**Proof.** It is sufficient to show that $\mathcal{B}$ is a filter base that is invariant under homotheties of strict positive ratio, and then to invoke **Proposition 4.5**.

*Invariance by homothety.* Let $B$ an absorbing, balanced, and convex set such that $B\cap\mathfrak{X}_n$ is a neighborhood of the origin in $\mathfrak{X}_n$ (for every $n\in\mathbb{N}$). For any $\alpha>0$ the set $\alpha B$ is still absorbing, balanced, and convex. On the other hand, $\alpha B\cap\mathfrak{X}_n = \alpha(B\cap\mathfrak{X}_n) \in \mathcal{V}_{\mathfrak{X}_n}(0)$, because $\mathcal{V}_{\mathfrak{X}_n}(0)$ is invariant under homothetic transformations of strictly positive ratio.

*Filter base.* Let $B'$ and $B''$ be two convex sets belonging to $\mathcal{B}$. The intersection $B = B'\cap B''$ is still absorbing, balanced, and convex (cf. **Proposition 1.26**). Also

$$B\cap\mathfrak{X}_n = (B'\cap\mathfrak{X}_n)\cap(B''\cap\mathfrak{X}_n) \in \mathcal{V}_{\mathfrak{X}_n}(0)$$

because $\mathcal{V}_{\mathfrak{X}_n}(0)$ is stable under finite intersection. The assertion follows. ■

### 5.2.1. A criterion for a convex subset to be a neighborhood of the origin in the strict inductive limit topology

**5.12. Proposition.** *Let $V$ be a* **convex** *subset of* $(\mathfrak{X}, \mathcal{V})$. *The following two assertions are equivalent:*

*i.* *$V$ is a neighborhood of the origin (in $\mathfrak{X}$), i.e., $V\in\mathcal{V}(0)$.*

*ii.* *For every $n\in\mathbb{N}$, the trace $V\cap\mathfrak{X}_n$ of $V$ on $\mathfrak{X}_n$ is a (convex) neighborhood of the origin in $\mathfrak{X}_n$. In other words, $V\cap\mathfrak{X}_n \in \mathcal{V}_n(0)$ $\forall n\in\mathbb{N}$.*

*Here, the notation is the same as the main section: $(\mathfrak{X}, \mathcal{V})$ is the strict inductive limit of the sequence $(\mathfrak{X}_n, \mathcal{V}_n)_{n\in\mathbb{N}}$; $\mathcal{V}$ and $\mathcal{V}_n$ are, respectively, the neighborhood topologies of $\mathfrak{X}$ and $\mathfrak{X}_n$.*

**Proof.** **[*i.*⇒*ii.*]** For this implication, the hypothesis that $V$ is convex does not play an essential role. Let $V$ be a set in $\mathcal{V}(0)$. Since $\mathfrak{X}$ is a locally convex space, the set $V$ contains a convex, balanced, and absorbing set $B\in\mathcal{V}(0)$, and to be a neighborhood of the origin in $\mathfrak{X}$ means that $B\cap\mathfrak{X}_n \in \mathcal{V}_n(0)$ for every $n\in\mathbb{N}$. But then, for every $n\in\mathbb{N}$, we have that also $V\cap\mathfrak{X}_n (\supseteq B\cap\mathfrak{X}_n)$ is a neighborhood of the origin in $\mathfrak{X}_n$.

**[*ii.*⇒*i.*]** Let $V$ be a convex set in $\mathfrak{X}$ such that, for every $n \in \mathbb{N}$, $V_n := V \cap \mathfrak{X}_n$ is a neighborhood of $0$ in $\mathfrak{X}_n$. The set $V_n$ is convex and contains a balanced set $E_n \in \mathcal{V}_n(0)$. We set $E = \bigcup_{n\in\mathbb{N}} E_n$. Clearly, $E$ is a balanced set — it is simple to prove that the union of a family of balanced sets is still balanced. But then the convex hull $K(E)$, of $E$ in $\mathfrak{X}$, is balanced and it is also in $\mathcal{V}(0)$, because $K(E)$ is convex, balanced, absorbing and $K(E) \cap \mathfrak{X}_n \supseteq E_n$ for every $n \in \mathbb{N}$. Obviously, $K(E) \subseteq V$ because $E \subseteq V$ and $V$ is convex (the fact that $E \subseteq V$ is a consequence of $E_n \subseteq V_n$ $\forall n \in \mathbb{N}$). Hence $V$ is a neighborhood of the origin in $\mathfrak{X}$. ■■■■

### 5.2.2. Characterization of continuous linear maps taking values in a locally convex space

**5.13. Proposition.** *Let $(\mathfrak{X}, \mathcal{V})$ be the strict inductive limit of the sequence of locally convex spaces $(\mathfrak{X}_n, \mathcal{V})_{n\in\mathbb{N}}$.*

**Assumptions:** *Assume that $f$ is a* **linear** *map from $\mathfrak{X}$ into $\mathfrak{Y}$, with $\mathfrak{Y}$ a locally convex space. For every $n \in \mathbb{N}$, denote by $f_n$ the* **restriction** *of $f$ to $\mathfrak{X}_n$.*

**Claim:** *The linear map $f$ is* **continuous** *if, and only if, for every $n \in \mathbb{N}$ the restriction $f_n$ is continuous from $\mathfrak{X}_n$ to $\mathfrak{Y}$.*

**PROOF.** Let $V$ be a neighborhood of $0$ in $\mathfrak{Y}$. We have to show that for every $n \in \mathbb{N}$ there exists $U_n \in \mathcal{V}_n(0)$ such that $f_n(U_n) \subseteq V$. For that, we observe that if $f$ is continuous, then $f^{-1}(V)$ is a neighborhood of $0$ in $\mathfrak{X}$ and, therefore, it contains a convex neighborhood $U$ of $0$ in $\mathfrak{X}$. For every $n \in \mathbb{N}$, $U_n := U \cap \mathfrak{X}_n$ is a neighborhood of $0$ in $\mathfrak{X}_n$ (due to **Proposition 5.12**) and, moreover,

$$U_n \subseteq f^{-1}(V) \cap \mathfrak{X}_n = f_n^{-1}(V) \equiv \{x \in \mathfrak{X}_n :: f_n(x) \in V\}.$$

Therefore, $f_n(U_n) \subseteq V$, and this shows that $f_n$ is continuous at $0$, hence everywhere (because $f_n$ is linear). Note that we didn't make use of the local convexity of $\mathfrak{Y}$.

For the other direction, suppose that for every $n \in \mathbb{N}$ the restriction $f_n$ is continuous (on $\mathfrak{X}_n$). Let $V \in \mathcal{V}_{\mathfrak{Y}}(0)$. Since $\mathfrak{Y}$ is **locally convex**, we can assume that $V$ is convex. Hence, $f^{-1}(V)$ is convex because $f$ is **linear** (cf. **Proposition 1.29**). Now, for every $n \in \mathbb{N}$ we have

$$f^{-1}(V) \cap \mathfrak{X}_n = f_n^{-1}(V),$$

and this is a neighborhood of $0$ in $\mathfrak{X}_n$ (by hypothesis). Due to **Proposition 5.12**, $f^{-1}(V)$ is a neighborhood of $0$ in $\mathfrak{X}$. ■■■■

In particular, the following result holds.

**5.14. Corollary.** *A linear form on $\mathfrak{X}$ is* **continuous** *if, and only if, its restriction to every $\mathfrak{X}_n$ is continuous.*

### 5.2.3. The topologies induced by the strict inductive limit on its generating subspaces

**5.15. Proposition.** *The topology $\mathcal{V}_n$ of $\mathfrak{X}_n$ coincides with the topology induced by $\mathcal{V}$ on $\mathfrak{X}_n$:*

$$\mathcal{V}_n \equiv \mathcal{V}|\mathfrak{X}_n.$$

*In other terms, for every $n \in \mathbb{N}$:*

$$\mathcal{V}_n \equiv \{V_n \subseteq \mathfrak{X}_n :: V_n = V \cap \mathfrak{X}_n \ \textbf{for some}\ V \in \mathcal{V}(0)\}.$$

**Proof.** $[\mathcal{V}_n \supseteq \mathcal{V}|\mathfrak{X}_n]$ We have to prove that every neighborhood $V_n$ of the origin in the induced topology $(\mathcal{V}|\mathfrak{X}_n)(0)$ contains a neighborhood $U_n \in \mathcal{V}_n(0)$.

By definition of induced topology, one has $V_n = V \cap \mathfrak{X}_n$ for some $V \in \mathcal{V}(0)$. But $\mathcal{V}$ is locally convex, hence $U \subseteq V$ for some *convex* $U \in \mathcal{V}(0)$. According to **Proposition 5.12**, for every $n \in \mathbb{N}$, $U_n := U \cap \mathfrak{X}_n$ is in $\mathcal{V}_n(0)$. Therefore, also $V_n \in \mathcal{V}_n(0)$ because it includes the neighborhood $U_n$. This completes the proof of the inclusion $\mathcal{V}|\mathfrak{X}_n \subseteq \mathcal{V}_n$.

The inclusion $\mathcal{V}_n \supseteq \mathcal{V}|\mathfrak{X}_n$ is also a consequence of **Proposition 5.13**. Indeed, the identity map $x \in (\mathfrak{X}, \mathcal{V}) \mapsto x \in (\mathfrak{X}, \mathcal{V})$ is linear and continuous, and the inclusion $\mathcal{V}_n \supseteq \mathcal{V}|\mathfrak{X}_n$ is nothing but the continuity of the immersion $x \in (\mathfrak{X}_n, \mathcal{V}_n) \hookrightarrow x \in (\mathfrak{X}, \mathcal{V})$.

$[\mathcal{V}|\mathfrak{X}_n \supseteq \mathcal{V}_n]$ Here we have to use, and for the first time, the property $\mathcal{V}_n \equiv \mathcal{V}_{n+1}|\mathfrak{X}_n$, given in the assumptions at the beginning of **Section 20**.

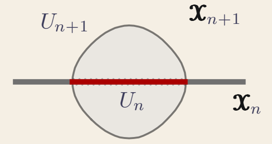



**Figure 5.4.** Since $\mathcal{V}_n = \mathcal{V}_{n+1}|\mathfrak{X}_n$, according to **Lemma 5.4**, there exists a convex neighborhood $U_{n+1} \in \mathcal{V}_{n+1}(0)$ such that $U_n = U_{n+1} \cap \mathfrak{X}_n$

Fix $n \in \mathbb{N}$. Let $U_n$ be a convex neighborhood in $\mathcal{V}_n(0)$. We have to show the existence of a neighborhood $U \in \mathcal{V}(0)$ such that $U \cap \mathfrak{X}_n \subseteq U_n$. Note that we are allowed to (and we will) construct $U$ depending on the fixed $n$, although we do not explicitly report this into the notation. Due to **Proposition 5.12**, it is sufficient to construct a *convex* set $U \subseteq \mathfrak{X}$ such that

Without loss of generality, we can consider just the convex neighborhoods. Indeed, they suffice to form a fundamental system of neighborhoods.

$$U \cap \mathfrak{X}_n \subseteq U_n \quad \textbf{and} \quad \forall k \in \mathbb{N}\left[U \cap \mathfrak{X}_k \in \mathcal{V}_k(0)\right]. \tag{5.6}$$

Since $\mathcal{V}_n = \mathcal{V}_{n+1}|\mathfrak{X}_n$, according to **Lemma 5.4**, there exists a convex neighborhood $U_{n+1} \in \mathcal{V}_{n+1}(0)$, such that $U_n \equiv U_{n+1} \cap \mathfrak{X}_n$. For the same reason, there exists a convex neighborhood $U_{n+2} \in \mathcal{V}_{n+2}(0)$, such that $U_{n+1} \equiv U_{n+2} \cap \mathfrak{X}_{n+1}$. Hence, $U_n \equiv U_{n+2} \cap \mathfrak{X}_{n+1} \cap \mathfrak{X}_n = U_{n+2} \cap \mathfrak{X}_n$. By induction, one shows the existence, for every $k \in \mathbb{N}$, of a convex neighborhood $U_{n+k} \in \mathcal{V}_{n+k}(0)$ such that $U_n \equiv U_{n+k} \cap \mathfrak{X}_n$. We set

$U_{n+1} \cap \mathfrak{X}_n = U_n$, $U_{n+2} \cap \mathfrak{X}_{n+1} = U_{n+1}, \ldots,$ $U_{n+k+1} \cap \mathfrak{X}_{n+k} = U_{n+k}$

$$U := \bigcup_{k \in \mathbb{N}} U_{n+k}.$$

Note that the set $U$ so constructed satisfies (5.6). Indeed, by construction, $U_{n+k} \subseteq U_{n+k+1}$ for every $k \in \mathbb{N}$, and this implies that $U$ is *convex* because it is the union of an increasing sequence of convex sets. Also, we have $U \cap \mathfrak{X}_n = \bigcup_{k \in \mathbb{N}} (U_{n+k} \cap \mathfrak{X}_n) = \bigcup_{k \in \mathbb{N}} U_n = U_n$, so that we got more than $U \cap \mathfrak{X}_n \subseteq U_n$ in (5.6). Finally, for every $k \in \mathbb{N}$ we have $U \cap \mathfrak{X}_k = \bigcup_{j \in \mathbb{N}} (U_{n+j} \cap \mathfrak{X}_k) \supseteq U_{n+k} \cap \mathfrak{X}_k$ and this last set is a neighborhood in $\mathcal{V}_k(0)$ because from $\mathcal{V}_n \equiv \mathcal{V}_{n+1}|\mathfrak{X}_n$ it follows that $\mathcal{V}_k \equiv \mathcal{V}_{n+k}|\mathfrak{X}_k$ for every $k \in \mathbb{N}$. ■

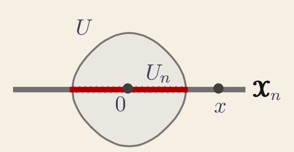



**Figure 5.5.** Let $x$ be a non zero element of $\mathfrak{X}$. Since $\mathcal{V}_n$ is Hausdorff separated, there exists a neighborhood $U_n$ of $0$ (in $\mathfrak{X}_n$) not passing through $x$.

**5.16. Corollary.** *If the topologies $\mathcal{V}_n$ are Hausdorff separated, then $\mathcal{V}$ is Hausdorff separated as well.*

**Proof.** Let $x$ be a non zero element of $\mathfrak{X}$. As $\mathfrak{X} = \bigcup_{n \in \mathbb{N}} \mathfrak{X}_n$, there exists an $n \in \mathbb{N}$ such that

$x \in \mathfrak{X}_n$. Since $\mathcal{V}_n$ is Hausdorff separated, there exists a neighborhood $U_n \in \mathcal{V}_n(0)$ not passing through $x$. Since $\mathfrak{X}_n$ is endowed with the topology induced by $\mathfrak{X}$, there exists a $U \in \mathcal{V}(0)$ such that $U_n = U \cap \mathfrak{X}_n$. Hence, $x \notin U$ (because otherwise $x \in U_n$ as $x \in \mathfrak{X}_n$), and this proves that $\mathfrak{X}$ is Hausdorff separated.

## 5.3 Strict inductive limit of Fréchet spaces (LF-spaces)

We call **LF-space** every **strict inductive limit of Fréchet spaces**. It is possible to show that, in general, an LF-space is not a Fréchet space. The argument behind this claim is based on the fact that every Fréchet space is metrizable (**Theorem 4.54**), and, on the other hand, there are examples of LF spaces that are not metrizable.

**Example 5.17. (A non-metrizable LF space)** The most important example (for us) of nonmetrizable LF-space is $\mathfrak{D}(\Omega)$ (if you don't know yet how convergence in $\mathfrak{D}(\Omega)$ is defined, come back to this observation later). To show that the topology of $\mathfrak{D}(\Omega)$ is not metrizable (cf. **Remark 6.27**), we use a reductio to absurdum argument. We focus on the case $\Omega = \mathbb{R}$, but the idea can be easily generalized (cf. **Remark 6.27**). Let $(\varphi_n)_n$ be a sequence in $\mathfrak{D}(\mathbb{R})$ with $\varphi_n(x) = 1$ for $|x| \leqslant n$ and $\varphi_n(x) = 0$ for $|x| > n+1$. Assume $d$ to be a metric on $\mathfrak{D}(\mathbb{R})$ that induces the same topology of $\mathfrak{D}(\mathbb{R})$. Let $B_n$ be the ball around $0$ with radius $1/n$ in this metric. As each $B_n$ is a neighborhood of the origin, it is absorbing. Thus, for each $n \in \mathbb{N}$, there exists $c_n > 0$ such that $\psi_n := c_n \varphi_n \in B_n$. But then, $\psi_n \to 0$ in $\mathfrak{D}(\mathbb{R})$ and this cannot be the case because $\mathrm{supp}_{\mathbb{R}} \psi_n = \mathrm{supp}_{\mathbb{R}} \varphi_n$ and there exists no compact subset of $\mathbb{R}$ which includes the support of every $\psi_n$ (the sequence $(\varphi_n)_{n \in \mathbb{N}}$ has been built to escape from every compact subset of $\mathbb{R}$).

### 5.3.1. Characterization of continuous linear maps in LF spaces

**5.18. Proposition. Assumption:** *Let $\mathfrak{X}$ be an* LF *space, $\mathfrak{Y}$ a topological vector space, and $f : \mathfrak{X} \to \mathfrak{Y}$ a* **linear** *map.* **Claim:** *If $\mathfrak{Y}$ is a locally convex space and $f$ is sequentially continuous, then $f$ is continuous.*

○

**PROOF.** The sequential continuity of $f$ on $\mathfrak{X}$ ensures the sequential continuity of the restriction $f_{|\mathfrak{X}_n}$ (on $\mathfrak{X}_n$). But each $\mathfrak{X}_n$, being a Fréchet space, has a countable filter base of neighborhoods of the origin and, therefore, sequential continuity on $\mathfrak{X}_n$ is equivalent to continuity on $\mathfrak{X}_n$. Thus, $\forall n \in \mathbb{N}$, the restriction $f_{|\mathfrak{X}_n}$ is continuous (on $\mathfrak{X}_n$). By **Proposition 5.13**, it follows that $f$ is continuous. This completes the proof.

### 5.3.2. Characterization of bounded subsets in LF spaces [Dieudonné-Schwartz]

**5.19. Theorem. (Dieudonné-Schwartz) Assumptions:** *Let $\mathfrak{X}$ be the* LF*-space defined by the family of Fréchet spaces $(\mathfrak{X}_n, \tau_n)_{n \in \mathbb{N}}$. Let $A$ be a* **bounded** *subset of $\mathfrak{X}$.* **Claim:** *There exists and $\nu \in \mathbb{N}$ such that $A \subseteq \mathfrak{X}_\nu$ (and therefore $A$ is also included in $\mathfrak{X}_n$ for every $n \geqslant \nu$, because of $\mathfrak{X}_n \trianglelefteq \mathfrak{X}_{n+1}$ $\forall n \in \mathbb{N}$).*

**PROOF.** We make use of the completeness assumption on the spaces $\mathfrak{X}_n$. We argue by logical contraposition: we prove that if there is no $\mathfrak{X}_\nu$, $\nu \in \mathbb{N}$, containing the set $A$ then $A$ is **un**bounded.

The argument proceeds as follows. If $A$ is not contained in any of the $\mathfrak{X}_n$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $A$ such that $x_n \notin \mathfrak{X}_n$ for every $n \in \mathbb{N}$. This allows us to *build a neighborhood* $U \in \mathcal{V}(0)$

Note that if $x_n \notin \mathfrak{X}_n$, then $\lambda x_n \notin \mathfrak{X}_n$ for every $\lambda \neq 0$.

*in* $\mathfrak{X}$ *which does not absorb the set* $S := \{x_n\}_{n\in\mathbb{N}}$. This will complete the proof because if $U \in \mathcal{V}(0)$ does not absorb $S$, then *it is not the case that* $S$ is absorbed by any neighborhood of the origin, that is, $S$ is unbounded. Since $A$ contains the unbounded set $S$, also $A$ is unbounded.

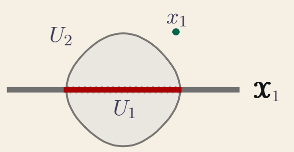



**Figure 5.6.** **Step 1.** Since $x_1 \notin \mathfrak{X}_1$ and $\mathfrak{X}_1$ is closed in $\mathfrak{X}_2$, there exists a convex neighborhood $U_2$ of $0$ in $\mathfrak{X}_2$ such that $U_1 = U_2 \cap \mathfrak{X}_1$ and $x_1 \notin U_2$.

After that, it is sufficient to prove that if the sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathfrak{X}$ is such that $x_n \notin \mathfrak{X}_n$ for every $n \in \mathbb{N}$, then $S := \{x_n\}_{n\in\mathbb{N}}$ is unbounded. First, we note that, by assumption, also $\frac{1}{n}x_n \notin \mathfrak{X}_n$ for every $n \in \mathbb{N}$. Then we observe what follows. It is possible to construct an increasing sequence $(U_n)_{n\in\mathbb{N}}$ of subsets of $\mathfrak{X}$ such that for every $n \in \mathbb{N}$ the set $U_n$ is a **convex neighborhood** of the origin in $\mathfrak{X}_n$ and, moreover,

Note that, for such a family, we have $U_n = U_{n+1} \cap \mathfrak{X}_n = U_{n+2} \cap \mathfrak{X}_{n+1} \cap \mathfrak{X}_n = U_{n+2} \cap \mathfrak{X}_n$, so that, in general, for every $k > 0$ we have $U_n = U_{n+k} \cap \mathfrak{X}_n$.

$$U_n = U_{n+1} \cap \mathfrak{X}_n \quad \textbf{and} \quad x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n \notin U_{n+1}. \tag{5.7}$$

This follows from **Corollary 5.8**.

**Initial step. Construction of $U_2$.** The construction starts from any convex neighborhood $U_1 \in \mathcal{V}_1(0)$. Then, by **Lemma 5.6**, since $x_1 \notin \mathfrak{X}_1$, $U_1 \subseteq \mathfrak{X}_1$, and $\mathfrak{X}_1$ is closed[5.1] in $\mathfrak{X}_2$, there exists a convex neighborhood $V_2 \in \mathcal{V}_2(0)$ such that (cf. **Lemma 5.6**)

$$U_1 = V_2 \cap \mathfrak{X}_1 \quad \textbf{and} \quad x_1 \notin V_2. \tag{5.8}$$

We set $U_2 := V_2$.

**Construction of $U_3$.** We look for a set $U_3 \subseteq \mathfrak{X}_3$ such that

$$U_2 = U_3 \cap \mathfrak{X}_2 \quad \textbf{and} \quad x_1, \frac{1}{2}x_2 \notin U_3. \tag{5.9}$$

Note that, if such a set $U_3$ exists, then $U_1 = U_2 \cap \mathfrak{X}_1 = U_3 \cap \mathfrak{X}_1$. Therefore $U_3$ must depend both on $U_2$ and $U_1$.

Since $\frac{1}{2}x_2 \notin \mathfrak{X}_2$ and $\mathfrak{X}_2$ is closed in $\mathfrak{X}_3$, there exists a convex neighborhood $V_3$ in $\mathcal{V}_3(0)$ such that (cf. **Lemma 5.6**)

$$U_2 = V_3 \cap \mathfrak{X}_2 \quad \text{and} \quad \frac{1}{2}x_2 \notin V_3.$$

The possible issue here is that we cannot set $U_3 := V_3$ because we do not know whether $x_1 \notin V_3$. We remedy by using **Corollary 5.8**. Since $x_1 \notin U_2 \subseteq \mathfrak{X}_2$ and $\mathfrak{X}_2$ is closed in $\mathfrak{X}_3$, there exists a convex neighborhood $W_3$ in $\mathcal{V}_3(0)$ such that (use again **Lemma 5.6**)

$$U_2 = W_3 \cap \mathfrak{X}_2 \quad \text{and} \quad x_1 \notin W_3.$$

Setting $U_3 := V_3 \cap W_3$, we conclude. Summarizing, up to now, we have $U_1 \subseteq U_2 \subseteq U_3$ and $x_1, \frac{1}{2}x_2 \notin U_3 = U_1 \cup U_2 \cup U_3$.

5.1. The topology $\mathcal{V}_1$ coincides with the topology induced on $\mathfrak{X}_1$ by the Hausdorff separated topology $\mathcal{V}_2$. But $\mathfrak{X}_1$ (endowed with $\mathcal{V}_1$) is *complete*, and therefore, according to **Proposition 3.33**, closed in $\mathfrak{X}_2$. **Reminder: Proposition 3.33.*iii*** states that *in a* **Hausdorff** *topological vector space, every complete subset is closed.*

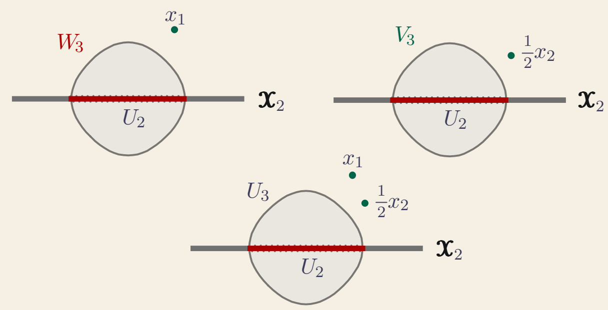



**Figure 5.7. Step 2.** Since $\frac{1}{2}x_2 \notin \mathfrak{X}_2$ and $\mathfrak{X}_2$ is closed in $\mathfrak{X}_3$, there exists a convex neighborhood $V_3$ of $0$ in $\mathfrak{X}_3$ such that $U_2 = V_3 \cap \mathfrak{X}_2$ and $\frac{1}{2}x_2 \notin V_3$. Setting $U_3 := V_3 \cap W_3$, we build a set $U_3$ such that $U_2 = U_3 \cap \mathfrak{X}_2$ and $x_1, \frac{1}{2}x_2 \notin U_3$. Note that, in principle, $x_1$ can belong to $\mathfrak{X}_2$, but in the picture what matters is that $x_1 \notin W_3$.

**Construction of $U_n$.** By induction on $n$, one can construct a sequence $(U_n)_{n\in\mathbb{N}}$ such that, roughly speaking, for every $n \in \mathbb{N}$ the set $U_{n+1}$ "*escapes*" from all the first $n$ elements of the sequence $((1/n)x_n)_{n\in\mathbb{N}}$ and $U_{n+1} \cap \mathfrak{X}_n = U_n$. Precisely, for every $n \in \mathbb{N}$ the set $U_n$ is a **convex neighborhood** of $0$ in $\mathfrak{X}_n$ and there holds

Note that the sequence of neighborhoods $(U_n)_{n\in\mathbb{N}}$ escapes in a strange way, that is by increasing its size. Indeed $U_n \subseteq U_{n+1}$ for every $n \in \mathbb{N}$. This is possible because, for every $n \in \mathbb{N}$, the point $x_n$ lies outside $\mathfrak{X}_n$ and therefore also outside $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_{n-1}$ owing to the fact that $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ is increasing.

$$U_n = U_{n+1} \cap \mathfrak{X}_n \quad \textbf{and} \quad x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n \notin U_{n+1}. \tag{5.10}$$

We then set

$$U := \bigcup_{n\in\mathbb{N}} U_n. \tag{5.11}$$

The set $U$ so built is a neighborhood of the origin (in $\mathfrak{X}$). Indeed, it is *convex* because union of an *increasing* sequence of convex sets; moreover, for every $n \in \mathbb{N}$, $U \cap \mathfrak{X}_n$ is in $\mathcal{V}_n(0)$ as it contains $U_n \equiv U_n \cap \mathfrak{X}_n \in \mathcal{V}_n(0)$.

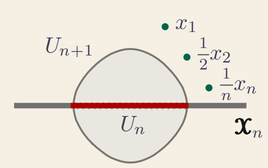



**Figure 5.8. Step 3.** By induction on $n$, one can construct a sequence $(U_n)_{n\in\mathbb{N}}$ such that: ▶ $U_n$ is a **convex neighborhood** of $0$ in $\mathfrak{X}_n$. ▶ $U_n = U_{n+1} \cap \mathfrak{X}_n$. ▶ $x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n \notin U_{n+1}$.

It remains to show that $U$ **cannot** absorb the set $S = \{x_n\}_{n\in\mathbb{N}}$. Indeed, if $U$ absorbs $S$, there exists $k \in \mathbb{N}$ such that $kU \supseteq S$. In particular, one has $\frac{1}{k}x_k \in U$ and, since $U := \bigcup_{n\in\mathbb{N}} U_n$ with $(U_n)_{n\in\mathbb{N}}$ increasing, there exists $\nu \in \mathbb{N}$ such that,

$$\frac{1}{k}x_k \in U_n \quad \textbf{for all } n \geqslant \nu.$$

But this cannot be the case because, by construction, for every $k \in \mathbb{N}$ there exists an *arbitrary* large $n \in \mathbb{N}$ such that $\frac{1}{k}x_k \notin U_n$ (precisely for every $n \geqslant k+1$).

**5.20. Corollary. Assumptions:** *Let $\mathfrak{X}$ be the* LF*-space defined by the family of Fréchet spaces $(\mathfrak{X}_n, \tau_n)_{n\in\mathbb{N}}$. A subset $A \subseteq \mathfrak{X}$ is a* ***bounded*** *subset of $\mathfrak{X}$* ***if, and only if****, there exists and $\nu \in \mathbb{N}$ such that $A \subseteq \mathfrak{X}_\nu$ and $A$ is bounded in $\mathfrak{X}_\nu$.*

Note that, strictly speaking, the assertion that $A$ is bounded in $\mathfrak{X}_\nu$ makes sense only when $A \subseteq \mathfrak{X}_\nu$. Therefore, formally, what is meaningful is the existence of a $\nu \in \mathbb{N}$ such that $A$ is bounded in $\mathfrak{X}_\nu$.

**Proof.** It is a direct consequence of Dieudonné-Schwartz theorem, together with the compatibility condition $\tau_\nu \equiv \tau|\mathfrak{X}_\nu$ and **Proposition 3.43**, according to which if $\mathfrak{M}$ is a topological vector **sub**space of the topological vector space $\mathfrak{X}$, a subset $A \subseteq \mathfrak{M}$ is bounded in $\mathfrak{X}$ **if**, **and only if**, it is bounded in $\mathfrak{M}$.

*We talk about topological vector* ***sub****space when the subspace is endowed with the subspace topology indced by* $\mathfrak{M}$

**5.21. Corollary.** *Let $\mathfrak{X}$ be the* LF*-space. A sequence $(x_n)_{n\in\mathbb{N}}$ converges in $\mathfrak{X}$ if, and only if, the following two assertions are satisfied:*

- ***i.*** *There exists $\nu \in \mathbb{N}^*$ such that the set $\{x_n\}_{n\in\mathbb{N}}$ is contained in $\mathfrak{X}_\nu$.*
- ***ii.*** *The sequence $(x_n)_{n\in\mathbb{N}}$ converges in $\mathfrak{X}_\nu$.*

*Note that, strictly speaking, assertion* ***ii.*** *makes sense only when the set $\{x_n\}_{n\in\mathbb{N}}$ is contained in $\mathfrak{X}_\nu$. Therefore, formally,* ***ii.*** *is a more general result.*

**Proof.** That **[*i.* and *ii.*]** imply the convergence of $(x_n)_{n\in\mathbb{N}}$ is a consequence of **Proposition 2.23** because, according to **Proposition 5.15**, $\mathcal{V}_\nu \equiv \mathcal{V}|\mathfrak{X}_\nu$. Note that this implication holds even for a generalized sequence.

The fact that the convergence of $(x_n)_{n\in\mathbb{N}}$ implies **[*i.* and *ii.*]** is a consequence of **Proposition 3.44** (point ***v.***) which guarantees that every Cauchy sequence is bounded. Indeed, as $(x_n)_{n\in\mathbb{N}}$ is bounded in $\mathfrak{X}$, from Dieudonné-Schwartz theorem, we infer assertion ***i.*** For ***ii.***, simply recall that $\mathfrak{X}_\nu$ is complete.

### 5.3.3. Completeness in LF-spaces

It is possible to prove the following result, whose proof can be found in [Wilansky, Albert. *Modern methods in topological vector spaces*. Courier Corporation, **2013**, **Theorem 13.3.13**].

**5.22. Proposition.** *Let $(\mathfrak{X}, \mathcal{V})$ be the strict inductive limit of the sequence of locally convex spaces $(\mathfrak{X}_n, \mathcal{V}_n)$. If for every $n \in \mathbb{N}$ the space $\mathfrak{X}_n$ is complete, then the inductive limit $\mathfrak{X}$ is complete.*

*In particular, any inductive limit of a sequence of Frechét spaces is complete (although, in general, the limit itself is not a Fréchet space).*

**5.23. Remark.** As we will see later, the previous proposition implies that the space $\mathfrak{D}(\Omega)$ is complete. However, it is not a Fréchet space because it is not metrizable (cf. **Example 5.17**). It follows that $\mathfrak{D}(\Omega)$ is not first countable (otherwise, it would be a Fréchet space). In particular, it is not second countable.

# 6

## Fundamental function spaces

### 6.1 Spaces of continuous functions and Radon measures

Given a topological space $\Omega$, the set $C(\Omega, \mathbb{K})$ of all continuous (but not necessarily bounded) functions defined on $\Omega$ is a (pure *algebraic*) **vector space** when endowed with the natural laws of addition among functions and multiplication by a scalar:

$$+ :: (f, g) \in C(\Omega, \mathbb{K}) \mapsto f + g := x \in \Omega \mapsto f(x) + g(x) \in \mathbb{K} \tag{6.1}$$
$$\cdot :: (\alpha, f) \in \mathbb{K} \times C(\Omega, \mathbb{K}) \mapsto \alpha f := x \in \Omega \mapsto \alpha f(x) \in \mathbb{K}. \tag{6.2}$$

Recall Definition 4.8. Let $\mathfrak{X}$ be a vector space and $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ a family of seminorms on $\mathfrak{X}$. ▶ We say that the family $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ is **total** (or **separating** or that it **separates points**) if for every $x \in \mathfrak{X}$ **different from zero** there exists an $\alpha(x) \in \mathcal{A}$, depending on $x$, such that $\mathfrak{p}_{\alpha(x)}(x) \neq 0$. Equivalently, the family $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ separate the points if whenever $\mathfrak{p}_\alpha(x) = 0$ holds for every $\alpha \in \mathcal{A}$ then necessarily $x = 0$. ▶ We say that the family $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ is **directed** or **filtering** if the **ordered set** $(\{\mathfrak{p}_\alpha\}_{\alpha \in \mathcal{A}}, \succcurlyeq)$, with $\succcurlyeq$ the usual order relation defined by $p_\alpha \succcurlyeq p_\beta$ **if and only if** $p_\alpha(x) \geqslant p_\beta(x)$ $\forall x \in \mathfrak{X}$, is a **directed set**. In other words, the family of seminorms $(\mathfrak{p}_\alpha)_{\alpha \in \mathcal{A}}$ is directed if, and only if, for every couple of seminorms $\mathfrak{p}_{\alpha_1}$ and $\mathfrak{p}_{\alpha_2}$ there exists always a seminorm $\mathfrak{p}_\alpha$ upper bounding them: $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_{\alpha_1}$ and $\mathfrak{p}_\alpha \succcurlyeq \mathfrak{p}_{\alpha_2}$. Note that the index set $\mathcal{A}$ is **not** assumed to be directed in general.

**Notation 6.1.** *Often, to shorten notation, we shall write $C(\Omega)$ instead of $C(\Omega, \mathbb{K})$.*

It is well-known that if $\Omega$ is a compact set, then the functional $\|\cdot\|_\infty$: $f \in C(\Omega) \to \mathfrak{p}_\Omega(f) := \sup_{x \in \Omega} f$ defines a norm on $C(\Omega)$, and the resulting normed space $(C(\Omega), \|\cdot\|_\infty)$ is even a Banach space. However, if $\Omega$ is not compact, there is no natural way to structure $C(\Omega, \mathbb{K})$ into a *normed* vector space. Instead, it is natural to endow $C(\Omega, \mathbb{K})$ with a topology compatible with the vector space structure as soon as certain compactness assumptions on the topological space $\Omega$ are made. Precisely, we assume that $\Omega$ is a **$\sigma$-locally compact Hausdorff** space. Under this hypothesis on $\Omega$, according to Proposition ?, there exists an increasing sequence $(\Omega_j)_{j \in \mathbb{N}}$, of open and relatively compact sets, covering $\Omega$ and such that $\bar{\Omega}_j \subseteq \Omega_{j+1}$ for every $j \in \mathbb{N}$. This property, as we are going to show, permits to supply $C(\Omega, \mathbb{K})$ with the structure of a locally convex space.

#### 6.1.1. The space $\mathfrak{C}(\Omega)$ with $\Omega$ a $\sigma$-locally compact Hausdorff space

Let $\Omega$ be a **$\sigma$-locally compact Hausdorff** space. We denote by $\mathfrak{K}_\Omega$ the family of all compact subsets of $\Omega$. For any $f \in C(\Omega)$ and any compact subset $K \in \mathfrak{K}_\Omega$, we set

$$\mathfrak{p}_K(f) := \sup_{x \in K} |f(x)| = \max_{x \in K} |f(x)|. \tag{6.3}$$

The following result holds.

**6.2. Proposition.** *As $K$ varies over all compact subsets of $\Omega$, the family $(\mathfrak{p}_K)_{K \in \mathfrak{K}_\Omega}$ describes a* filtering *and* total (separating) *family of seminorms on $C(\Omega)$.*

Recall that *total* and *separating* are synonyms

**Proof.** For any $K \in \mathfrak{K}_\Omega$ the functional $\mathfrak{p}_K$ is clearly a seminorm. The family $(\mathfrak{p}_K)_{K \in \mathfrak{K}_\Omega}$ is filtering because if $K = K_1 \cup K_2$, with $K_1, K_2 \in \mathfrak{K}_\Omega$, then $\mathfrak{p}_{K_1} \vee \mathfrak{p}_{K_2} \leqslant \mathfrak{p}_K$ (actually, $\mathfrak{p}_{K_1} \vee \mathfrak{p}_{K_2} = \mathfrak{p}_K$). The family is separating (total) because the singletons are compact and $\mathfrak{p}_{\{a\}}(f) = |f(a)|$ for every $a \in \Omega$. ■

The **natural** topology on $C(\Omega, \mathbb{K})$ is the **locally convex** topology $\tau_\Omega$ defined by this family of seminorms (in the sense of Proposition 4.15 and Definition 4.16). The topological space $(C(\Omega, \mathbb{K}), \tau_\Omega)$

is then denoted by the symbol $\mathfrak{C}(\Omega, \mathbb{K})$, or by the symbol $\mathfrak{C}(\Omega)$ if it is clear from the context the underlying scalar field $\mathbb{K}$. The topology $\tau_\Omega$ is called the topology of **uniform convergence on all compact subsets** of $\Omega$. Indeed, as a consequence of **Proposition 4.34**, the convergence of $(f_\lambda)_{\lambda\in\Lambda}$ to $f$, in $\mathfrak{C}(\Omega)$, is equivalent to the condition $\lim_\Lambda \mathfrak{p}_K(f_\lambda - f) = 0$, for every compact subset $K \in \mathfrak{K}_\Omega$.

**More details.** Let us particularize **Proposition 4.15** to the algebraic vector space $C(\Omega)$. As $(\mathfrak{p}_K)_{K\in\mathfrak{K}_\Omega}$ is a **filtering** family of seminorms on the (purely) **algebraic** vector space $C(\Omega)$, we have that:

We stress that, at this preliminary stage, it makes no sense to wonder if these seminorms are continuous. It will make sense only after Claim $i$.

**Claim *i*.** It is possible to structure $X$ into a locally convex space declaring as a fundamental system of neighborhoods of the origin the set consisting of all possible closed semiballs (of any strictly positive «radius») of the seminorms of the family. In other words, we define a filter basis $\mathcal{B}$ on $C(\Omega)$ by setting

$$\begin{aligned}\mathcal{B} := \{\rho B_\bullet(\mathfrak{p}_K)\}_{(\rho,K)\in\mathbb{R}_+^*\times\mathfrak{K}_\Omega} &= \{\{f\in C(\Omega) :: \mathfrak{p}_K(f)\leqslant\rho\}\}_{(\rho,K)\in\mathbb{R}_+^*\times\mathfrak{K}_\Omega}\\ &= \{\{f\in C(\Omega) :: \sup_{x\in K}|f(x)|\leqslant\rho\}\}_{(\rho,K)\in\mathbb{R}_+^*\times\mathfrak{K}_\Omega}.\end{aligned}$$

**Claim *ii*.** Every seminorm $\mathfrak{p}_K$ is then **continuous** on $C(\Omega)$ with respect to this topology $\tau_\Omega$ (generated by $\mathcal{B}$) and therefore, for every $K\in\mathfrak{K}_\Omega$ we have $(B_\bullet(\mathfrak{p}_K))^\circ = B_\circ(\mathfrak{p}_K)$ and $B_\bullet(\mathfrak{p}_K) = \overline{B_\circ(\mathfrak{p}_K)}$.

**Claim *iii*.** The locally convex topology $\tau_\Omega$ is (Hausdorff) separated because the family $(\mathfrak{p}_K)_{K\in\mathfrak{K}_\Omega}$ separates the points.

Up to now, we did not use the hypothesis that $\Omega$ is a $\sigma$-locally compact Hausdorff space. The assumption is used in the next result, which shows that the space $\mathfrak{C}(\Omega)$ admits *a countable basis of continuous seminorms*. In particular, cf. **Theorem 4.56**, $\mathfrak{C}(\Omega)$ is a pre-Fréchet space. Later on, we show that $\mathfrak{C}(\Omega)$ is also complete and, therefore, a Fréchet space (cf. **Theorem 6.46**).

**6.3. Proposition.** *The space* $\mathfrak{C}(\Omega)$ *admits* a countable basis of continuous seminorms*. Therefore, due to* **Proposition 4.45***,* $\mathfrak{C}(\Omega)$ *is a pre-Fréchet space.*

**PROOF.** It is sufficient to take, as a basis of continuous seminorms, the one associated with the countable family of domains $(K_j := \bar{\Omega}_j)_{j\in\mathbb{N}}$ where $(\Omega_j)_{j\in\mathbb{N}}$ is an exhaustion of $\Omega$ by open and relatively compact sets (cf. **Remark ?**).

Let us fill in the details. By definition, the family $(\mathfrak{p}_{K_j})_{j\in\mathbb{N}}$ is a (countable) basis of continuous seminorms if, and only if, the family $\mathcal{B} := \{\rho B_\bullet(\mathfrak{p}_{K_j})\}_{(\rho,j)\in\mathbb{R}_+^*\times\mathbb{N}}$ is a fundamental system of neighborhoods of the origin. Therefore, we have to show that for every neighborhood $\rho B_\bullet(\mathfrak{p}_K)$ there exist $j_\star\in\mathbb{N}$ and $\rho_\star > 0$ such that $\rho_\star B_\bullet(\mathfrak{p}_{K_j}) \subseteq \rho B_\bullet(\mathfrak{p}_K)$. For that, is is sufficient to show that for every $B_\bullet(\mathfrak{p}_K)$ there exists $j_\star\in\mathbb{N}$ such that $B_\bullet(\mathfrak{p}_{K_j}) \subseteq B_\bullet(\mathfrak{p}_K)$.

This last statement is simple to prove. Indeed, since $(K_j := \bar{\Omega}_j)_{j\in\mathbb{N}}$ is an exhaustion of $\Omega$, there exists[6.1] $j_\star\in\mathbb{N}$ such that $K\subseteq K_j$ for every $j\geqslant j_\star$. Hence $B_\bullet(\mathfrak{p}_{K_j})\subseteq B_\bullet(\mathfrak{p}_K)$ for every $j\geqslant j_\star$, and this concludes the proof.

Recall that if $K_1, K_2\in\mathfrak{K}_\Omega$ and $K_1\subseteq K_2$, then $\mathfrak{p}_{K_1}\leqslant\mathfrak{p}_{K_2}$ and therefore $B_\bullet(\mathfrak{p}_{K_2})\subseteq B_\bullet(\mathfrak{p}_{K_1})$.

An alternative argument relies on the use of **Corollary 4.31** which, when specialized to this context, reads as follows: $(\mathfrak{p}_{K_j})_{j\in\mathbb{N}}$ is a (countable) basis of continuous seminorms on $\mathfrak{C}(\Omega)$ **if, and only if**, for every continuous seminorm $\mathfrak{p}\in(\mathfrak{p}_K)_{K\in\mathfrak{K}_\Omega}$, there exists a seminorm $\mathfrak{q}\in(\mathfrak{p}_{K_j})_{j\in\mathbb{N}}$ and a constant $c_\mathfrak{p} > 0$ such that $\mathfrak{p}(\varphi)\leqslant c_\mathfrak{p}\,\mathfrak{q}(\varphi)$ for every $\varphi\in\mathfrak{C}(\Omega)$. But then, it is sufficient to observe that $\mathfrak{p}_K(\varphi)\leqslant\mathfrak{p}_{K_j}(\varphi)$ for every $j\geqslant j_\star$. ■■■■

**6.4. Remark.** It is possible to show (cf. **Theorem 6.46**) that $\mathfrak{C}(\Omega)$ is a complete space (i.e., that the topology $\tau_\Omega$ turns the space $C(\Omega)$ into a complete space). Therefore, $\mathfrak{C}(\Omega)$ *is a Fréchet space*. Note, however, that the completeness does not follow from **Proposition 5.22**, because $\tau_\Omega$ is not the inductive limit of Frechet spaces: it has been built from a family of *seminorms*.

6.1. Let $K$ be a compact subset of $\Omega$. Since $(\Omega_j)_{j\in\mathbb{N}}$ covers $K$, it is possible to extract a finite family $\{\Omega_{j_1}, \Omega_{j_2}, \ldots, \Omega_{j_\star}\}$ with $j_1\leqslant j_2\leqslant\ldots\leqslant j_\star$ which still covers $K$. As $(\Omega_j)_{j\in\mathbb{N}}$ is increasing we have $K\subseteq\Omega_{j_\star}$ and therefore $K\subseteq\Omega_j$ for every $j\geqslant j_\star$.

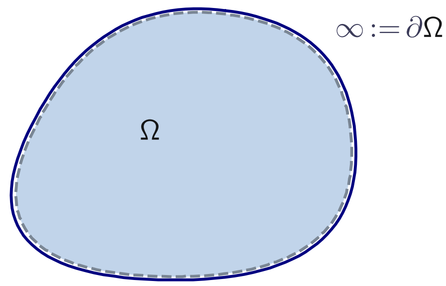



**Figure 6.1.** When $\Omega$ is a bounded open set of $\mathbb{R}^N$, the Alexandrov compactification of $\Omega$ can be thought as obtained by considering on $\dot{\Omega} = \Omega \oplus \{\infty\}$ the topology of $\bar{\Omega} = \Omega \oplus \partial\Omega$ with the identification $\infty := \partial\Omega$.

**Example 6.5.** Let $\Omega$ be any open set of $\mathbb{R}^N$, then the characteristic function $\chi_\Omega$ of $\Omega$ is in $\mathfrak{C}(\Omega)$. More generally, the restriction to $\Omega$ of any continuous function defined on $\mathbb{R}^N$ is in $\mathfrak{C}(\Omega)$. If $\Omega$ is the open unit ball of $\mathbb{R}^N$, the function $x \mapsto \exp(-1/(1-|x|^2))$ is in $\mathfrak{C}(\Omega)$. If $\Omega$ is the punctured unit ball of of $\mathbb{R}^N$, that is the set $\{x \in \mathbb{R}^N :: 0 < |x| < 1\}$, then the function $x \mapsto |x|^{-\alpha}$, $\alpha \in \mathbb{R}$, is in $\mathfrak{C}(\Omega)$. In fact, elements of $\mathfrak{C}(\Omega)$ need not to be bounded functions.

### 6.1.2. The space $\mathfrak{C}_0(\Omega)$ with $\Omega$ a $\sigma$-locally compact Hausdorff space

We denote by $C_0(\Omega)$ the subset of $C(\Omega)$ consisting of all those continuous functions that **vanish at infinity**. This is the vector subspace of $C(\Omega)$ consisting of continuous functions such that

$$\forall \varepsilon > 0 \, \exists K_\varepsilon \in \mathfrak{K}_\Omega :: |f(x)| < \varepsilon \quad \forall x \in \Omega \setminus K_\varepsilon. \tag{6.4}$$

**6.6. Remark.** The terminology comes from the fact that, as it is possible to show, if $\dot{\Omega} = \Omega \cup \{\infty\}$ is the Alexandrov compactification of the locally compact space $\Omega$, obtained by adjoining the point at infinity, then the space $C_0(\Omega)$ coincides with all continuous functions on $C(\Omega)$ that can be extended to $\dot{\Omega}$ by assigning the value zero at the point $\infty$.

Recall that if we denote by $\tau$ the topology on $\Omega$, then the Alexandrov topology on $\dot{\Omega}$ is defined by

$$\dot{\tau} := \tau \oplus \{(\Omega \setminus K) \oplus \{\infty\} :: K \in \mathfrak{K}_\Omega\}.$$

where we use the symbol '$\oplus$' as a substitute of '$\cup$' just to emphasize that the union is disjoint. It follows that if $\Omega$ is **compact**, then $\dot{\Omega} = \Omega \oplus \{\infty\}$ with $\Omega$ and $\{\infty\} = (\Omega \setminus \Omega) \oplus \{\infty\}$ both open and disjoints. Thus, when $\Omega$ is compact, the Alexandrov compactification gives a disconnected set and is not very interesting in this case.

Also, if $\Omega$ is compact, then $C(\Omega) = C_0(\Omega)$. Indeed, if $\Omega$ is compact, then for every $\varepsilon > 0$ one can take $K_\varepsilon := \Omega$ to make vacously true the statement (6.4). From an Alexandrov perspective, the essence of the story is that when $\Omega$ is compact, the elements of $C_0(\Omega)$ that vanish at infinity are elements of $C(\Omega)$ subject to the fictitious constraint that they have to vanish on the connected component consisting of the single point at infinity $\{\infty\}$.

A necessary and sufficient condition for an element of $f \in C(\Omega)$ to be in $C_0(\Omega)$, i.e., *vanishing at infinity*, is that for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \Omega$, such that $|f(x)| < \varepsilon$ for every $x \in \Omega \setminus K_\varepsilon$. With this in mind, let us show the following equivalence.

**6.7. Proposition.** *The space $C_0(\Omega)$ coincides with the space of all continuous functions $f \in C(\Omega)$ such that for every $\varepsilon > 0$ the set $\{x \in \Omega :: |f(x)| \geqslant \varepsilon\}$ is compact.*

**6.8. Remark.** Proposition 6.7 gives another proof of the equality $C_0(\Omega) = C(\Omega)$ when $\Omega$ is compact. Indeed, if $\Omega$ is compact, then for every $\varepsilon > 0$ the set $\{x \in \Omega :: |f(x)| \geqslant \varepsilon\}$ is a closed subset of the compact set $\Omega$ and, therefore, compact.

**Proof.** Indeed, let $\varepsilon > 0$. Assuming $f \in C_0(\Omega)$, there exists a compact subset $K_\varepsilon \subset \Omega$ such that $|f(x)| < \varepsilon$ for every $x \in \Omega \backslash K_\varepsilon$. Thus $\{|f| \geqslant \varepsilon\} \subseteq K_\varepsilon$. But $\{|f| \geqslant \varepsilon\}$ is a closed subset (included in the compact set $K_\varepsilon$) and, therefore, by weak inheritance, compact[6.2].

On the other hand, suppose that for every $\varepsilon > 0$ the set $K_\varepsilon := \{x \in \Omega :: |f(x)| \geqslant \varepsilon\}$ is compact. Then, $|f| < \varepsilon$ in $\Omega \setminus K_\varepsilon$. This officially completes the proof. However, we want to remark that if $\Omega \setminus K_\varepsilon = \emptyset$, then $\Omega$ is compact and $|f| \geqslant \varepsilon$ in $\Omega$. ■

**6.9. Corollary.** *A continuous function $f: \Omega \to \mathbb{K}$, i.e., an element of $C(\Omega)$, belongs to $C_0(\Omega)$ if, and only if, there exists a sequence of compact sets $(K_j)_{j \in \mathbb{N}}$ such that*

$$\lim_{j \to \infty} \left( \sup_{x \in \Omega \backslash K_j} |f(x)| \right) = 0. \tag{6.5}$$

*More specifically, in one direction, the following (apparently) stronger implication holds. If $f \in C_0(\Omega)$ then (6.5) holds for every sequence of domains $(K_j := \bar{\Omega}_j)_{j \in \mathbb{N}}$ such that $(\Omega_j)_{j \in \mathbb{N}}$ is an exhaustion of $\Omega$ by open and relatively compact sets.*

**6.10. Remark.** In (6.5), the understanding is that the supremum is computed in $\mathbb{R}_+$, i.e., that $\sup \emptyset = 0$. In this way, if $\Omega$ is compact, one can take the constant sequence $K_j := \Omega$ to infer that for every $f \in C(\Omega)$ one has $\lim_{j \to \infty} (\sup_{x \in \Omega \backslash K_j} |f(x)|) = 0$. Also, note that if $\Omega$ is compact and $(\Omega_j)_{j \in \mathbb{N}}$ is an exhaustion of $\Omega$ by open and relatively compact sets, then $\Omega$ necessarily belongs to the exhaustion. Indeed, $(\Omega_j)_{j \in \mathbb{N}}$ covers the compact set $\Omega$ and, therefore, we can extract a finite cover. Since the sequence $(\Omega_j)_{j \in \mathbb{N}}$ is increasing, the extracted finite subcover has to contain $\Omega$ among its elements. In particular, the sequence $(\Omega_j)_{j \in \mathbb{N}}$ is eventually constant when $\Omega$ is compact, i.e., $\Omega_j = \Omega$ except for a finite number of terms.

**Proof.** Assume that (6.5) holds. This trivially implies that for every $\varepsilon > 0$, there exists $\nu(\varepsilon) \in \mathbb{N}$ such that $\sup_{x \in \Omega \backslash K_{\nu(\varepsilon)}} |f(x)| < \varepsilon$. In fact, (6.5) implies that $\sup_{x \in \Omega \backslash K_j} |f(x)| < \varepsilon$ for every $j \geqslant \nu(\varepsilon)$). On the other hand, assume $f \in C_0(\Omega)$. Since $\Omega$ is a $\sigma$-locally compact Hausdorff space, there exists a sequence of domains $(K_j := \bar{\Omega}_j)_{j \in \mathbb{N}}$ such that $(\Omega_j)_{j \in \mathbb{N}}$ is an exhaustion of $\Omega$ by open and relatively compact sets. Any exhaustion does the job. Indeed, for every $\varepsilon > 0$ there exists an element $\nu(\varepsilon) \in \mathbb{N}$ such that $\sup_{x \in \Omega \backslash K_{\nu(\varepsilon)}} |f(x)| < \varepsilon$. Since $(K_j := \bar{\Omega}_j)_{j \in \mathbb{N}}$ is increasing, we have $\sup_{x \in \Omega \backslash K_j} |f(x)| < \varepsilon$ for every $j \geqslant \nu(\varepsilon)$; this is nothing but (6.5). ■

Let us also prove the following simple yet useful observation in the more specific context of $\mathbb{R}^N$.

**6.11. Proposition.** *If $\Omega$ is a bounded open subset of $\mathbb{R}^N$, then $C_0(\Omega)$ consists of the restrictions to $\Omega$ of all continuous functions in $\mathbb{R}^N$ that are identically zero on $\partial\Omega$.*

**Proof.** Let $f \in C_0(\Omega)$. If we denote by $f\chi_\Omega$ the extension by zero of $f$ to $\mathbb{R}^N$ then $f$ is continuous in $\mathbb{R}^N$, i.e., $f\chi_\Omega \in C(\mathbb{R}^N)$. Indeed, $f$ is continuous at any point of $\mathbb{R}^N \backslash \partial\Omega$. Therefore it remains to show the continuity on $\partial\Omega$. Let $x \in \partial\Omega$, and let $(x_\lambda)_{\lambda \in \Lambda}$ be a generalized sequence converging to

6.2. The property of being compact is weakly hereditary. Recall that a topological space property is called weakly hereditary if whenever a topological space has that property, so does any closed subspace. In contrast, a topological space property is called hereditary if whenever a topological space has that property, so does any subspace.

$x$. For every $\varepsilon>0$ there exists a compact subset $K_\varepsilon\subseteq\Omega$ such that $|f|<\varepsilon$ on $\Omega\backslash K_\varepsilon$. Moreover, as $(x_\lambda)_{\lambda\in\Lambda}\to x$, there exists $\nu_\varepsilon\in\Lambda$ such that $x_\lambda\in(\Omega\backslash K_\varepsilon)\cup(\mathbb{R}^N\backslash\Omega)=(\mathbb{R}^N\backslash K_\varepsilon)$ for every $\lambda>\nu_\varepsilon$. Hence

$$\begin{aligned}|f(x_\lambda)|&<\varepsilon \quad\text{for every}\quad \lambda>\nu_\varepsilon \text{ such that } x_\lambda\in\Omega\backslash K_\varepsilon,\\ |f(x_\lambda)|&=0 \quad\text{for every}\quad \lambda>\nu_\varepsilon \text{ such that } x_\lambda\in\mathbb{R}^N\backslash\Omega.\end{aligned}$$

Overall, we have $|f(x_\lambda)|<\varepsilon$ for every $\lambda>\nu_\varepsilon$. Thus $(f(x_\lambda))_{\lambda\in\Lambda}\to 0$. By the arbitrariness of the generalized sequence $(f(x_\lambda))_{\lambda\in\Lambda}$ we conclude.

Vice versa, assume that $f\in C(\mathbb{R}^N)$ is such that $f_{|\partial\Omega}\equiv 0$, and let us prove that $f_{|\Omega}\in C_0(\Omega)$. We use the characterization of $C_0(\Omega)$ given in **Proposition 6.7**. For every $\varepsilon>0$

$$\{|f_{|\Omega}(x)|\geqslant\varepsilon\}=\{|f_{|\bar{\Omega}}(x)|\geqslant\varepsilon\}. \tag{6.6}$$

Indeed $\bar{\Omega}=\Omega\cup\partial\Omega$ and $f<\varepsilon$ on $\partial\Omega$ (actually, by assumption, $f\equiv 0$ on $\partial\Omega$). But $\{|f_{|\bar{\Omega}}(x)|\geqslant\varepsilon\}=\bar{\Omega}\cap\{|f|\geqslant\varepsilon\}$ is the intersection of a compact subset of $\mathbb{R}^N$ (the set $\bar{\Omega}$) and a closed subset of $\mathbb{R}^N$ (the set $\{|f|\geqslant\varepsilon\}$). Hence $\{|f_{|\bar{\Omega}}(x)|\geqslant\varepsilon\}$ is compact, and therefore so is $\{|f_{|\Omega}(x)|\geqslant\varepsilon\}$ by (6.6). ■■■■

Although the space $C_0(\Omega)$ can be seen as a locally convex subspace of $\mathfrak{C}(\Omega)$, in general, it is not a closed subset of $\mathfrak{C}(\Omega)$ (in fact, cf. **Proposition 6.30**, the closure of $C_0(\Omega)$ in $\mathfrak{C}(\Omega)$ is the whole of $\mathfrak{C}(\Omega)$). Therefore, $C_0(\Omega)$ is not complete[6.3] with the topology induced by $\mathfrak{C}(\Omega)$. That is the reason why one usually endow the space $C_0(\Omega)$ with the norm

$$\|f\|_\infty:=\sup_{x\in\Omega}|f(x)|. \tag{6.7}$$

Indeed, cf. **Proposition 6.13**, with the supremum norm the space $C_0(\Omega)$ becomes a complete normed space. We then set $\mathfrak{C}_0(\Omega):=(C_0(\Omega),\|\cdot\|_\infty)$. Note that the supremum in (6.7) is a maximum. Indeed, if $\Omega$ is compact, or if $f$ is identically zero, the assertion is trivial. Instead, if $|f(x_0)|>0$ for some $x_0\in\Omega$, then, by **Corollary 6.9**, there exists a compact subset $K\in\mathfrak{K}_\Omega$ passing through $x_0$ such that $|f(x)|<|f(x_0)|$ for every $x\in\Omega\setminus K$. Therefore

$$\sup_{x\in\Omega}|f(x)|=\sup_{x\in K}|f(x)|=\max_{x\in K}|f(x)|. \tag{6.8}$$

In particular, every element of $C_0(\Omega)$ is bounded. In fact, $C_0(\Omega)\subseteq C_b(\Omega)\subseteq C(\Omega)$ with $C_b(\Omega)$ consisting of those continuous functions in $C(\Omega)$ which are *bounded* in $\Omega$.

Clearly, the following assertion holds.

**6.12. Proposition.** *If $\Omega$ is compact, then $\mathfrak{C}_0(\Omega)=\mathfrak{C}(\Omega)$.*

It is simple to show that $\mathfrak{C}_0(\Omega)$ is a Banach space and the topology generated by $\|\cdot\|_\infty$ on $C_0(\Omega)$ is called the *topology of the uniform convergence on* $\Omega$.

**6.13. Proposition.** *$\mathfrak{C}_0(\Omega)$ is a Banach space.*

**Proof.** It is well known that the set $C_b(\Omega,\mathbb{K})$ consisting of those continuous functions in $C(\Omega,\mathbb{K})$ which are *bounded* in $\Omega$ is a vector subspace of $C(\Omega,\mathbb{K})$, and that $\mathfrak{C}_b(\Omega):=(C_b(\Omega,\mathbb{K}),\|\cdot\|_\infty)$ is a Banach space. Therefore, it is sufficient to prove that $C_0(\Omega)$ is a closed subspace of $\mathfrak{C}_b(\Omega)$.

6.3. It is well-known that subspaces of complete metric spaces are closed if, and only if, they are complete (see, e.g., ProofWiki). A similar result holds in Hausdorff separated topological vector spaces. A vector subspace of a complete and Hausdorff separated topological vector space is complete if, and only if, it is closed (see, e.g., pp. 47-51 and pp. 115-154 in Narici L., Beckenstein E., *Topological Vector Spaces*, Pure and applied mathematics, Boca Raton, FL: CRC Press (2011)).

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C_0(\Omega)$ such that $\|f_n - f\|_\infty \to 0$, for some $f \in \mathfrak{C}_b(\Omega)$. We have to show that $f \in C_0(\Omega)$. For that, consider a sequence of domains $(K_j := \bar{\Omega}_j)_{j\in\mathbb{N}}$ such that $(\Omega_j)_{i\in\mathbb{N}}$ is an exhaustion of $\Omega$ by open and relatively compact sets. Again, if $\Omega$ is compact, then $C_0(\Omega) = C_b(\Omega) = C(\Omega)$ and there is nothing to prove. By virtue of **Corollary 6.9**, it is sufficient to show that

$$\lim_{j\to\infty}\left(\sup_{x\in\Omega\setminus K_j}|f(x)|\right) = 0.$$

By assumption (and again by **Corollary 6.9**) for every $n \in \mathbb{N}$, we have $\lim_{j\to\infty}(\sup_{x\in\Omega\setminus K_j}|f_n(x)|) = 0$. Now, observe that

$$\sup_{x\in\Omega\setminus K_j}|f(x)| \leqslant \|f_n - f\|_\infty + \sup_{x\in\Omega\setminus K_j}|f_n(x)|.$$

Passing to the limit for $j \to \infty$ in both members of the previous relation, we get that for every $n \in \mathbb{N}$

$$\lim_{j\to\infty}\left(\sup_{x\in\Omega\setminus K_j}|f(x)|\right) \leqslant \|f_n - f\|_\infty.$$

Passing to the limit for $n \to \infty$ we conclude.

**Example 6.14.** Let $\Omega$ be any open set of $\mathbb{R}^N$, then the characteristic function $\chi_\Omega$ of $\Omega$ is **not** in $\mathfrak{C}_0(\Omega)$. If $\Omega$ is the open unit ball of $\mathbb{R}^N$, the function $x \mapsto \exp(-1/(1-|x|^2))$ is in $\mathfrak{C}_0(\Omega)$. If $\Omega$ is the *punctured* unit ball of $\mathbb{R}^N$, that is the set $\{x \in \mathbb{R}^N :: 0 < |x| < 1\}$, then the function $x \mapsto |x|^{-\alpha}$, $\alpha \in \mathbb{R}$, is **not** in $\mathfrak{C}_0(\Omega)$. The function $x \mapsto (1+|x|^\alpha)^{-\beta}$ is in $\mathfrak{C}_0(\mathbb{R}^N)$ for any $\alpha, \beta > 0$.

### 6.1.3. The support of a function

**6.15. Definition. (Support of a continuous function)** Let $f : \Omega \to \mathfrak{Y}$ be a function defined on a topological space $\Omega$ and with values in a topological vector space $\mathfrak{Y}$. The **domain of nullity** of $f$, is the biggest open subset $U_\Omega(f)$ of $\Omega$ where $f$ is identically zero. In other words,

$$U_\Omega(f) := \max_{(\subseteq)}\{U \subseteq \Omega :: U \text{ is open in } \Omega \textbf{ and } f_{|U} \equiv 0\} \tag{6.9}$$

$$= \bigcup\{U \subseteq \Omega :: U \text{ is open in } \Omega \textbf{ and } f_{|U} \equiv 0\}. \tag{6.10}$$

In other words, $U_\Omega(f)$ is the interior of the zero-level set of $f$:

$$U_\Omega(f) = \{x \in \Omega :: f(x) = 0\}^\circ. \tag{6.11}$$

**6.16. Remark.** If $f$ is a continuous then the zero-level set of $f$ is a closed subset and, therefore, $U_\Omega$, being the interior of a closed set, is an example of a regular open set (cf. **Definition ?**): $U_\Omega(f) = (\overline{U_\Omega(f)})^\circ$.

The complement $\Omega \setminus U_\Omega(f)$ of the domain of nullity is, by definition, the **support** of $f$ and is denoted by $\operatorname{supp}_\Omega f$. Clearly, $\operatorname{supp}_\Omega f$ is a closed set (because the complement of an open set). Moreover, from the relation $\Omega \setminus (E^\circ) = \overline{\Omega \setminus E}$ which is valid for any subset $E \subseteq \Omega$, we infer that the support of $f$ coincides with the closure (in $\Omega$) of the (open) set where $f$ is different from zero:

$$\begin{aligned}\operatorname{supp}_\Omega f &= \Omega \setminus U_\Omega(f)\\ &= \Omega \setminus (\{x \in \Omega :: f(x) = 0\}^\circ)\\ &= \overline{\Omega \setminus \{x \in \Omega :: f(x) = 0\}}\\ &= \overline{\{x \in \Omega :: f(x) \neq 0\}}.\end{aligned} \tag{6.12}$$

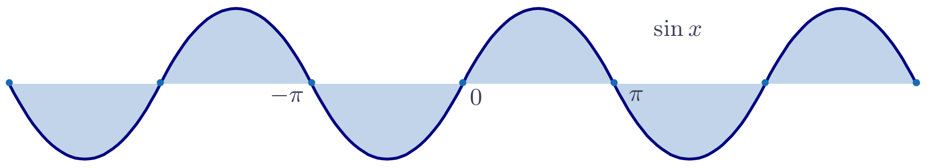



**Figure 6.2.** Although the function $\sin\colon x \in \mathbb{R} \mapsto \sin x$ takes the value zero on any point of the form $x_k = \pi k$ with $k \in \mathbb{Z}$, we have $\operatorname{supp}_{\mathbb{R}} \sin = \mathbb{R}$ because there is no open subset of $\mathbb{R}$ where $\sin$ is identically zero.

The closure, of course, is taken in $\Omega$.

**6.17. Proposition.** *Let $f\colon \Omega \to \mathcal{Y}$ be a function defined on a topological space $\Omega$ and with values in a topological vector space $\mathcal{Y}$. The following relation holds:*

$$(\operatorname{supp}_\Omega f)^\circ = \Omega \setminus \overline{U_\Omega(f)}.$$

*Therefore if $\operatorname{supp}_\Omega f$ has an empty interior then the set of zeros of $f$ is dense in $\Omega$. In particular, if $f$ is continuous in $\Omega$ and $\mathcal{Y}$ is Hausdorff separated, then $\operatorname{supp}_\Omega f$ has empty interior if, and only if, $f \equiv 0$ in $\Omega$.*

**Proof.** Recalling that the complement of the interior is the same as the closure of the complement, we have $\Omega \setminus (\operatorname{supp}_\Omega f)^\circ = \overline{\Omega \setminus \operatorname{supp}_\Omega f} = \overline{U_\Omega(f)}$ from which the first assertion follows (because the set of zeros of $f$ includes $U_\Omega(f)$). If $f$ is continuous and $\mathcal{Y}$ is Hausdorff separated, we can invoke the principle of extension of the identities (cf. **Proposition 2.39**) to conclude. ■■■■

**6.18. Remark.** The implication in **Proposition 6.17** cannot be reversed. In other words, it is not necessarily the case that if the set of zeros of $f$ is dense in $\Omega$ then $(\operatorname{supp}_\Omega f)^\circ = \emptyset$. For example, if $\chi_{\mathbb{Q}}\colon \mathbb{R} \to \mathbb{R}$ is the indicator function of the set of rational numbers (the so-called Dirichlet function), then set of zeros of $\chi_{\mathbb{Q}}$ is dense in $\mathbb{R}$ but $U_\Omega(\chi_{\mathbb{Q}}) = \emptyset$ and, therefore, $\operatorname{supp}_\Omega \chi_{\mathbb{Q}} = \mathbb{R}$.

**6.19. Remark.** In the definition of $\operatorname{supp}_\Omega f$, the closure of $\{x \in \Omega :: f(x) \neq 0\}$ must be taken in $\Omega$. Thus, the support of the constant function $\chi_{(0,1)}\colon x \in (0,1) \subseteq \mathbb{R} \mapsto 1 \in \mathbb{R}$ is $(0,1)$, while the support of the constant function $\chi_{[0,1]}\colon x \in [0,1] \subseteq \mathbb{R} \mapsto 1$ is $[0,1]$. Also, one does not have to confuse the support of $f$ with the set where $f$ is different from zero. For example, although the function $\sin\colon x \in \mathbb{R} \mapsto \sin x$ takes the value zero on any point of the form $x_k = \pi k$ with $k \in \mathbb{Z}$, we have $\operatorname{supp}_{\mathbb{R}} \sin = \mathbb{R}$ because there is no open subset of $\mathbb{R}$ where the sine function is identically zero (cf. **Figure 6.2**). Moreover, note that if $f\colon \Omega \to \mathbb{R}$ is a real analytic function defined in the open and connected subset $\Omega$ of $\mathbb{R}^N$ then $\operatorname{supp}_\Omega f = \Omega$ unless $f$ is the function identically equal to zero (in this case $\operatorname{supp}_\Omega f = \emptyset$). Indeed, by the identity theorem for real analytic functions, if $f$ vanishes on an open subset of $\Omega$ then $f$ vanishes everywhere in $\Omega$.

### 6.1.4. The space $\mathcal{K}(\Omega)$ with $\Omega$ a $\sigma$-locally compact Hausdorff space

Let $\Omega$ be a $\sigma$-locally compact topological space, and let $K$ be a **compact** subset of $\Omega$. We denote by $\mathcal{K}_K(\Omega) := (C_K(\Omega), \|\cdot\|_\infty)$ the (topological) **subspace** of $\mathcal{C}_0(\Omega)$ consisting of all those functions whose support (a posteriori necessarily compact) is contained in $K$. In other words

$$C_K(\Omega) := \{ f \in C_0(\Omega) :: \operatorname{supp}_\Omega f \subseteq K \}$$

and $\mathcal{K}_K(\Omega)$ is endowed with the topology induced by $\mathcal{C}_0(\Omega)$. Clearly, the space $\mathcal{K}_K(\Omega)$ **is a Banach space** (in fact, $C_K(\Omega)$ is a closed subset of $\mathcal{C}_0(\Omega)$).

Note that $\mathfrak{K}_K(\Omega)$ can also be seen as a topological vector subspace of $\mathfrak{C}(\Omega)$ because the topology induced by $\mathfrak{C}(\Omega)$ on $C_K(\Omega)$ coincide with the topology induce on $C_K(\Omega)$ by $\mathfrak{C}_0(\Omega)$. This easy yet important observation is formally stated later in **Proposition 6.29**.

We denote by $C_c(\Omega)$ the vector space consisting of all those continuous functions whose support is **compact** and contained in $\Omega$. Note that when $\Omega$ is compact, $C_c(\Omega) \equiv C_0(\Omega) \equiv C(\Omega)$. Again, according to **Proposition ?**, there exists an increasing sequence $(\Omega_j)_{j\in\mathbb{N}}$, of open and relatively compact sets, covering $\Omega$ and such that $\bar{\Omega}_j \subseteq \Omega_{j+1}$ for every $j \in \mathbb{N}$. Clearly, if we set $K_j := \bar{\Omega}_j$ we have

$$C_c(\Omega) = \cup_{j\in\mathbb{N}} C_{K_j}(\Omega). \tag{6.13}$$

Moreover, the following statement hold:

*i.* For every $j \in \mathbb{N}$ one has $C_{K_j}(\Omega) \trianglelefteq C_{K_{j+1}}(\Omega)$ and moreover $C_c(\Omega) \equiv \cup_{j\in\mathbb{N}} C_{K_j}(\Omega)$.

*ii.* The topology of $\mathfrak{K}_{K_j}(\Omega) = (C_{K_j}(\Omega), \tau_{K_j})$ coincides with the topology induced on $C_{K_j}(\Omega)$ by $\mathfrak{K}_{K_{j+1}}(\Omega) = (C_{K_{j+1}}(\Omega), \tau_{K_{j+1}})$, because both of them are inherited by $\mathfrak{C}(\Omega)$ (equivalently, by $\mathfrak{C}_0(\Omega)$).

Recall the *compatibility result* for induced topologies. In general, if $S$ is a topological space, and $A \subseteq B \subseteq S$, the subspace topology that $A$ inherits from the subspace $B$ (endowed with the topology induced by $S$) is the same as the one it inherits from $S$.

We are in the general setting of a strict inductive limit of Frechét spaces (cf. **Section 20**). In fact, every $\mathfrak{K}_{K_j}(\Omega) = (C_{K_j}(\Omega), \|\cdot\|_\infty)$ is a Banach space (in particular a Frechét space).

**6.20. Definition.** The natural topology on $C_c(\Omega)$ is, by definition, the strict inductive limit $\tau_{\text{LF}}$ of the topologies of Fréchet spaces $(\mathfrak{K}_{K_j}(\Omega))_{j\in\mathbb{N}} = (C_{K_j}(\Omega), \tau_{K_j})_{j\in\mathbb{N}}$. The resulting space

$$\mathfrak{K}(\Omega) := (C_c(\Omega), \tau_{\text{LF}})$$

is the **locally convex space** of those continuous functions whose support is compact and contained in $\Omega$.

To have a consistent definition, we have to show that limit topology on $\mathfrak{K}(\Omega)$ does not depend on the covering sequence $(\Omega_j)_{j\in\mathbb{N}}$. To this end, we observe that, according to **Proposition 5.15**, the topology of $\mathfrak{K}_{K_j}(\Omega)$ coincides with the topology induced on $C_{K_j}(\Omega)$ by $\mathfrak{K}(\Omega)$. Since every compact set $K \in \mathfrak{K}_\Omega$ is included in some $K_{j\star}$ of $(K_j := \bar{\Omega}_j)_{j\in\mathbb{N}}$, the following result holds.

**6.21. Proposition.** *For any $K \in \mathfrak{K}_\Omega$ the topology of $\mathfrak{K}_K(\Omega)$ coincides with the topology induced on $C_K(\Omega)$ by $\mathfrak{K}(\Omega)$.*

**PROOF.** The argument is purely topological and coincides with the one that is used later in the proof of **Proposition 6.56**. ■■■■

We then have the following immediate consequence.

**6.22. Corollary.** *The strict inductive limit topology on $\mathfrak{K}(\Omega)$ does not depend on the choice of the exhaustion $(\Omega_j)_{j\in\mathbb{N}}$.*

**PROOF.** The argument is purely topological and coincides with the one that is used later in the proof of **Corollary 6.57**. ■■■■

Moreover, as a direct consequence of **Proposition 5.13** and of **Proposition 5.18**, we get the following result.

**6.23. Proposition.** *For a linear map $u$ from $\mathcal{K}(\Omega)$ into a locally convex space, the following three assertions are equivalent:*

i. *The linear map $u$ is continuous.*

ii. *The linear map $u$ is sequentially continuous.*

iii. *For every compact set $K$ of $\Omega$, the restriction of $u$ to $\mathcal{K}_K(\Omega)$ is (sequentially) continuous.*

Convergence in $\mathcal{K}(\Omega)$ can be characterized by particularizing the Dieudonné-Schwartz theorem (**Theorem 5.19**), more precisely **Corollary 5.20** and **Corollary 5.21**, to the current setting.

**6.24. Proposition.** *Let $\Omega$ be a $\sigma$-locally compact topological space. A subset $B(\Omega)$ of $\mathcal{K}(\Omega)$ is bounded if, and only if, there exists a compact subset $K \in \mathfrak{K}_\Omega$ such that $B(\Omega) \subseteq \mathcal{K}_K(\Omega)$ and $B(\Omega)$ is bounded in $\mathcal{K}_K(\Omega)$. Namely:*

$$\big(\operatorname{supp}_\Omega \varphi \subseteq K \quad \forall \varphi \in B(\Omega)\big) \quad \textbf{and} \quad \left(\sup_{\varphi \in B(\Omega)} \mathfrak{p}_K(\varphi) < \infty\right).$$

*A sequence $(\varphi_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{K}(\Omega)$ converges in $\mathcal{K}(\Omega)$,* **if**, **and only if**:

i. *There exists a compact set $K \in \mathfrak{K}_\Omega$ such that the set $(\varphi_n)_{n\in\mathbb{N}}$ is contained in $\mathcal{K}_K(\Omega)$, that is, if $\operatorname{supp}_\Omega \varphi_n \subseteq K$ for every $n \in \mathbb{N}$.*

ii. *The sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{K}_K(\Omega)$.*

*Note that, under the two previous conditions, $\varphi_n \to \varphi$ in $\mathcal{K}(\Omega)$ for some $\varphi \in \mathcal{K}_K(\Omega)$ and, therefore, necessarily $\operatorname{supp}_\Omega \varphi \subseteq K$.*

Eventually, by **Proposition 5.22**, we infer the following result.

**6.25. Proposition.** *The space $\mathcal{K}(\Omega) := (C_c(\Omega), \tau_{\mathrm{LF}})$ is a* **complete locally convex space**.

**6.26. Remark.** When $\Omega$ is compact we have $\mathcal{K}(\Omega) \equiv \mathcal{C}(\Omega) \equiv \mathcal{C}_0(\Omega)$.

**6.27. Remark.** Note however, that if $\Omega$ is not compact, then $\mathcal{K}(\Omega)$ *is not a Frechet space*. In fact, it is not metrizable. In particular, $\mathcal{K}(\Omega)$ does not admit a countable basis of neighborhoods of the origin. To show that $\mathcal{K}(\Omega)$ is not metrizable, let us consider the usual sequence of domains $(K_j := \bar{\Omega}_j)_{j\in\mathbb{N}}$ with $(\Omega_j)_{j\in\mathbb{N}}$ an exhaustion of $\Omega$ by open and relatively compact sets. To the sequence $(K_j := \bar{\Omega}_j)_{j\in\mathbb{N}}$ we associate a sequence of Urysohn functions $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(\Omega)$, such that $\varphi_n(x) = 1$ for $x \in K_n$ and $\varphi_n(x) = 0$ for $x \in \Omega \backslash K_{n+1}$. Aiming at a reductio ad absurdum argument, assume that $d_{\mathrm{LF}}$ is a metric on $C_c(\Omega)$ which induces the same topology $\tau_{\mathrm{LF}}$ of $\mathcal{K}(\Omega)$. If the metric $d_{\mathrm{LF}}$ induces the topology $\tau_{\mathrm{LF}}$, then the $d_{\mathrm{LF}}$-ball $B_n(\Omega)$ centered at the origin and of radius $1/n$ has to be a neighborhood of the origin. In particular, $B_n(\Omega)$ has to be absorbing and, therefore, for each $n \in \mathbb{N}$ there exists $c_n > 0$ such that $\psi_n := c_n \varphi_n \in B_n(\Omega)$. But then, $d_{\mathrm{LF}}(\psi_n, 0) \to 0$ in $\mathbb{R}$, and therefore, $\psi_n \to 0$ in $\mathcal{K}(\Omega)$. But this is in contradiction with the characterization given in **Proposition 6.24**. Indeed, there exists no compact subset of $\Omega$ which includes the support of every $\psi_n$ because $\operatorname{supp}_\Omega \psi_n = \operatorname{supp}_\Omega \varphi_n$ and the sequence of Urysohn functions $(\varphi_n)_{n\in\mathbb{N}}$ has been built so that $\operatorname{supp}_\Omega \varphi_n$ escape from every compact subset of $\Omega$ (in fact, $\Omega = \bigcup_{n\in\mathbb{N}} \operatorname{supp}_\Omega \varphi_n$). Summarizing, if $\mathcal{K}(\Omega)$ is metrizable, then there exists a sequence of functions $(\psi_n)_{n\in\mathbb{N}}$ that converges to zero in $\mathcal{K}(\Omega)$ and, at the same time, does not converge in $\mathcal{K}(\Omega)$. This is, hopefully, a contradiction. ⋯

**Example 6.28.** Let $\Omega$ be any open set of $\mathbb{R}^N$, then the characteristic function of $\Omega$, $\chi_\Omega: x\in\Omega\mapsto 1\in\mathbb{R}$, is **not** in $\boldsymbol{\mathcal{K}}(\Omega)$. If $\Omega$ is the open unit ball of $\mathbb{R}^N$, the function $\eta: x\in\Omega\mapsto \exp(-1/(1-|x|^2))\in\mathbb{R}$ is in $\boldsymbol{\mathcal{C}}_0(\Omega)$ but not in $\boldsymbol{\mathcal{K}}(\Omega)$. On the other hand, $\eta\chi_\Omega\in\boldsymbol{\mathcal{K}}(\mathbb{R}^N)$. Note that if $\Omega$ is an open and connected subset of $\mathbb{R}^N$, there exists no real analytic function $f:\Omega\to\mathbb{R}$ in $\boldsymbol{\mathcal{K}}(\Omega)$ other then the identically zero function. Indeed, if $f$ is not identically equal to zero, then $\mathrm{supp}_\Omega f$ is a compact subset of $\Omega$ and, therefore, $U_\Omega(f)$ is an open subset of $\mathbb{R}^N$. By the identity theorem for real analytic functions, if $f$ vanishes on $\Omega$ then $f$ vanishes everywhere in $\Omega$.

### 6.1.5. Relations among the spaces $\boldsymbol{\mathcal{C}}(\Omega), \boldsymbol{\mathcal{C}}_0(\Omega), \boldsymbol{\mathcal{K}}_K(\Omega)$ and $\boldsymbol{\mathcal{K}}(\Omega)$

In literature, the space $\boldsymbol{\mathcal{K}}_K(\Omega)$ is often denoted as $C_K(\Omega)$, while the space $\boldsymbol{\mathcal{K}}(\Omega)$ is simply denoted as $C_c(\Omega)$. Instead, we shall stick on a conceptual distinction. For us, the (purely algebraic) vector spaces $C_K(\Omega)$ and $C_c(\Omega)$ are defined as the carrier sets of the locally convex spaces $\boldsymbol{\mathcal{K}}_K(\Omega)$ and $\boldsymbol{\mathcal{K}}(\Omega)$. In other words$\boldsymbol{\mathcal{C}}_0(\Omega)$

Recall that the carrier set of a topological space $(X,\tau)$ is simply $X$.

$$\boldsymbol{\mathcal{K}}_K(\Omega)=(C_K(\Omega),\|\cdot\|_\infty) \quad \textbf{and} \quad \boldsymbol{\mathcal{K}}(\Omega)=(C_c(\Omega),\tau_{\mathrm{LF}}).$$

**Order relation among the topologies.** Obviously, for any compact subset $K$ of $\Omega$, one has

| $(C_K(\Omega),\|\cdot\|_\infty)$ | $\subseteq$ | $(C_c(\Omega),\tau_{\mathrm{LF}})$ | $\subseteq$ | $(C_0(\Omega),\|\cdot\|_\infty)$ | $\subseteq$ | $(C(\Omega),\tau_\Omega)$ |
|---|---|---|---|---|---|---|
| $=$ | | $=$ | | $=$ | | $=$ |
| $\boldsymbol{\mathcal{K}}_K(\Omega)$ | | $\boldsymbol{\mathcal{K}}(\Omega)$ | | $\boldsymbol{\mathcal{C}}_0(\Omega)$ | | $\boldsymbol{\mathcal{C}}(\Omega)$ |

.

More precisely:

- The strict inductive limit topology on $\boldsymbol{\mathcal{K}}(\Omega)$ is finer than the topology induced on $C_c(\Omega)$ by $\boldsymbol{\mathcal{C}}_0(\Omega)$.

  This is trivial, but to realize this, observe that this assertion means that the inclusion $\boldsymbol{\mathcal{K}}(\Omega)\hookrightarrow (C_c(\Omega),\|\cdot\|_\infty)$ is continuous. As the inclusion is a linear map, and $(C_c(\Omega),\|\cdot\|_\infty)$ is a locally convex space (actually a normed space), to prove that the inclusion is continuous, one can check that the restriction of the inclusion to any $\boldsymbol{\mathcal{K}}_K(\Omega)$ is continuous. And now one can immediately realize that it is trivial because if $(x_n)_{n\in\mathbb{N}}\to 0$ in $\boldsymbol{\mathcal{K}}_K(\Omega)$ then clearly $(\|x_n\|_\infty)_{n\in\mathbb{N}}\to 0$, or, equivalently, because the topology of $\boldsymbol{\mathcal{K}}_K(\Omega)$ coincides with the one induced by $\boldsymbol{\mathcal{C}}_0(\Omega)$ on $C_K(\Omega)$.

- The topology of $\boldsymbol{\mathcal{C}}_0(\Omega)$ is finer than the topology induced on $C_0(\Omega)$ by $\boldsymbol{\mathcal{C}}(\Omega)$.

  This is trivial because if $(x_n)_{n\in\mathbb{N}}\to 0$ uniformly on $\Omega$, it converges uniformly on every compact subset of $\Omega$. To show that, in general, the topology is strictly finer, one can consider the sequence $(\varphi_n(x):=\varphi(x-n))_{n\in\mathbb{N}}$ in $\boldsymbol{\mathcal{C}}_0(\mathbb{R})$ defined by translating by $n\in\mathbb{N}$ a function $\varphi$ in $C_c(\mathbb{R})$ with nonempty support, e.g., a triangular pulse. Clearly, $\varphi_n$ converges to zero on every compact subset of $\mathbb{R}$, but $\varphi_n$ does not converge uniformly in $\mathbb{R}$ to zero. The counterexample could seem a consequence of having chosen an unbounded subset of $\mathbb{R}$. But this is not the case. A similar example can be constructed on the open interval $(0,1)$ as in **Figure 6.3**.

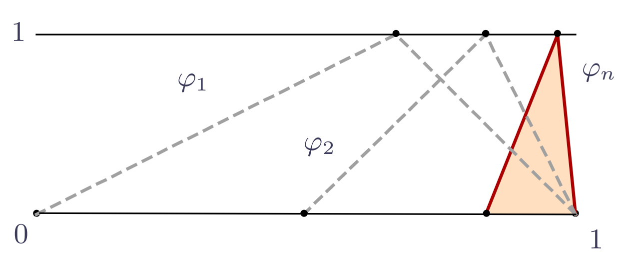



**Figure 6.3.** A sequence of function in $C_0(\Omega)$, with $\Omega=(0,1)$, which converges to zero uniformly on every compact subset of $\Omega$, but does not converge to zero uniformly on $\Omega$ (because $\sup_\Omega|\varphi_n|=1$ for every $n\in\mathbb{N}$).

It is also clear that

**6.29. Proposition.** *On $C_K(\Omega)$ the topologies induced by $\mathfrak{K}(\Omega)$, $\mathfrak{C}_0(\Omega)$ or $\mathfrak{C}(\Omega)$ are identical.*

**6.30. Proposition. (Density relations)** *The vector space $C_c(\Omega)$ is dense both in $\mathfrak{C}_0(\Omega)$ and $\mathfrak{C}(\Omega)$:*

$$\mathbf{Closure}_{(C_0(\Omega),\|\cdot\|_\infty)}[C_c(\Omega)] = \mathfrak{C}_0(\Omega) \quad \textit{and} \quad \mathbf{Closure}_{(C(\Omega),\tau_\Omega)}[C_c(\Omega)] = \mathfrak{C}(\Omega).$$

*In particular, since $C_c(\Omega) \subseteq C_0(\Omega)$ we also have* $\mathbf{Closure}_{(C(\Omega),\tau_\Omega)}[C_0(\Omega)] = \mathfrak{C}(\Omega)$.

These density relations imply (cf. **Remark 6.31**) that the space $C_c(\Omega)$ is not going to be complete as a topological vector subspace of $\mathfrak{C}(\Omega)$ or $\mathfrak{C}_0(\Omega)$, as well as that $C_0(\Omega)$ is not going to be complete as a topological vector subspace of $\mathfrak{C}(\Omega)$. Therefore, it is not a closed vector subspace of $\mathfrak{C}(\Omega)$ and, in particular, it is not complete with the topology induced by $\mathfrak{C}_0(\Omega)$.

**PROOF.** Let us prove that $\mathbf{Closure}_{(C_0(\Omega),\|\cdot\|_\infty)}[C_c(\Omega)] = \mathfrak{C}_0(\Omega)$. As usual, we can focus on the case in which $\Omega$ is not compact. Pick any element $f_0 \in \mathfrak{C}_0(\Omega)$. By definition, given $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subseteq \Omega$ such that

$$\sup_{x\in\Omega\setminus K_\varepsilon} |f_0(x)| < \varepsilon.$$

On the other hand, according to Urysohn lemma (**Lemma ?**), there exists a Urysohn function $\eta_\varepsilon \in C_c(\Omega)$, $0 \leqslant \eta_\varepsilon \leqslant 1$, such that $\eta_\varepsilon \equiv 1$ on $K_\varepsilon$. We set $f_\varepsilon := \eta_\varepsilon f_0$. Clearly $f_\varepsilon \in C_c(\Omega)$ and moreover, as $|f_0 - f_\varepsilon| = 0$ on $K_\varepsilon$, we have

$$\begin{aligned} \|f_0 - f_\varepsilon\|_\infty &= \max\left\{\sup_{x\in K_\varepsilon}|f_0 - f_\varepsilon|, \sup_{x\in\Omega\setminus K_\varepsilon}|f_0 - f_\varepsilon|\right\} \\ &= \sup_{x\in\Omega\setminus K_\varepsilon}|f_0 - f_\varepsilon| = \sup_{x\in\Omega\setminus K_\varepsilon}|1-\eta_\varepsilon|\,|f_0| \\ &= \sup_{x\in\Omega\setminus K_\varepsilon}|f_0| < \varepsilon. \end{aligned}$$

The arbitrariness of $\varepsilon$ concludes the proof.

**6.31. Remark.** Let us make a comment on the density of $C_c(\Omega)$ in $\mathfrak{C}_0(\Omega)$ and $\mathfrak{C}(\Omega)$. We want to point out a possible lack of completeness that other possible topological choices could have introduced, stressing in this way the reason why the topologies we introduced are somewhat *natural*.

First, note that the sup-norm $\|f\|_\infty := \sup_{x\in\Omega}|f(x)|$ is also a norm on $C_c(\Omega)$, but if endowed with this norm the space $C_c(\Omega)$ is *not* complete because of the density of $C_c(\Omega)$ in $\mathfrak{C}_0(\Omega)$. Indeed, the density of $C_c(\Omega)$ in $\mathfrak{C}_0(\Omega)$ means, in particular, that for every $f \in C_0(\Omega)\setminus C_c(\Omega)$ there exists a sequence of functions $f_n \in C_c(\Omega)$ which, although a Cauchy sequence, does not converge in the normed space $(C_c(\Omega), \|\cdot\|_\infty)$ because (by construction) it converges in $\mathfrak{C}_0(\Omega)$ to $f \notin C_c(\Omega)$. Since completeness is a fundamental requirement for function spaces, both in theory and in the applications, that is the reason why one endows the space $C_c(\Omega)$ with the topology $\tau_{\mathrm{LF}}$ that turns it into the **complete** space $\mathfrak{K}(\Omega)$.

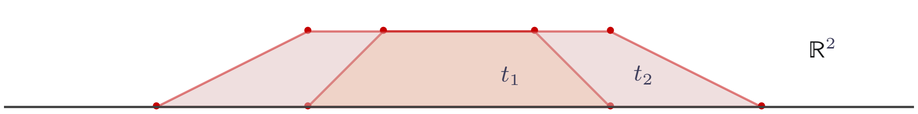



**Figure 6.4.** The sequence $t_n\chi_{[-n,n]}$ is in $C_c(\mathbb{R})$, but $t_n\chi_{[-n,n]} \to 1$ in $\mathfrak{C}(\mathbb{R})$. Indeed, $t_n\chi_{[-n,n]} \to 1$ uniformly on every compact subsets of $\mathbb{R}$.

In the same way, the topological vector space $(C_c(\Omega), \tau_\Omega)$, i.e., the vector space $C_c(\Omega)$ endowed with the $\mathfrak{C}(\Omega)$-topology of uniform convergence on all compact subsets of $\Omega$, is not complete. That is why one endows the space $C_c(\Omega)$ with a topology that turns it into the complete space $\mathfrak{K}(\Omega)$. The existence of a Cauchy sequence $(C_c(\Omega), \tau_\Omega)$ which does not converge is clear on a theoretical ground: one can follow the same argument described for $(C_c(\Omega), \|\cdot\|_\infty)$. Nevertheless, a concrete example is given by the sequence of trapezoidal functions $\chi_{[-n,n]}(x)t_n(x)$ with $t_n(x)$ having as graph in $\mathbb{R}^2$ the trapezoid (having the main base removed) of bases $[-n,n]\times\{0\}$ and $[-n/2,n/2]\times\{1\}$. The function $t_n(x)\chi_{[-n,n]}(x)$ is the extension by zero of $t_n$ outside $\chi_{[-n,n]}$. It is clear (cf. **Figure 6.4**) that $t_n\chi_{[-n,n]}\in C_c(\mathbb{R})$ and that $t_n\chi_{[-n,n]}\to 1$ in $\mathfrak{C}(\mathbb{R})$, because $t_n\chi_{[-n,n]}\to 1$ uniformly on every compact subsets of $\mathbb{R}$. But the constant function $x\in\mathbb{R}\mapsto 1$ is not in $C_c(\mathbb{R})$. ···

### 6.1.6. Radon measures

**6.32. Definition.** We call (complex) **Radon measure** on $\Omega$, every linear form which is *continuous on the topological vector space* $\mathfrak{K}(\Omega)$.

The value of a Radon measure $\mu$ on a function $\varphi\in\mathfrak{K}(\Omega)$ is usually denoted in one of the following ways:

$$\mu(\varphi), \quad \langle\mu,\varphi\rangle, \quad \int_\Omega \varphi\,\mathrm{d}\mu.$$

The set of all Radon measures on $\Omega$ is, therefore, nothing but the topological dual of $\mathfrak{K}(\Omega)$ and is denoted by $\mathfrak{K}'(\Omega)$ or by $\mathfrak{M}(\Omega)$. One will endow this space both with the strong-$*$ dual topology and the weak-$*$ dual topology. Note, however, that in this context, the weak-$*$ dual topology is usually referred to as the **vague topology** or as **the topology of vague convergence of measures**.

## 6.2 The spaces $\mathcal{E}^k(\Omega)$ ($k \in \bar{\mathbb{N}}, \Omega \subseteq \mathbb{R}^N, \Omega$ open)

The symbol $\bar{\mathbb{N}}$ stands for the extended set of natural numbers $\mathbb{N} \cup \{\infty\} = \{0, 1, 2, \ldots, \infty\}$.

### 6.2.1. Multi-index notation

We denote by $\Omega$ a nonempty open subset of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$). The generic point of $\Omega$ will be denoted by $x = (x_1, \ldots, x_N)$. The euclidean norm of $x$ will be denoted by $|x|$. In other terms, for every $x \in \mathbb{R}^N$ we set $|x|^2 := x_1^2 + \cdots + x_N^2$.

We shall make use of the **multi-index notation**. This is a convenient mathematical notation that simplifies formulas used in partial differential equations and the theory of distributions. It generalizes the concept of an integer index to an ordered tuple of indices.

**6.33. Definition.** An $N$-dimensional **multi-index** is an $N$-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{N}^N$ of non-negative integers.

For every multi-index $\alpha \in \mathbb{N}^N$ and $x \in \mathbb{R}^N$, we define the **absolute value** of $\alpha$ and the **monomial** $x^\alpha$ as

$$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_N \quad \textbf{and} \quad x^\alpha := x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \cdots \cdot x_N^{\alpha_N}. \tag{6.14}$$

Thus, $x^\alpha$ is a monomial of degree $|\alpha|$ in $N$ variables. The set $\mathbb{N}^N$ of multi-indices is endowed with an operation of addition and with a partial order relation. Precisely, for every pair of multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N), \beta = (\beta_1, \beta_2, \ldots, \beta_N) \in \mathbb{N}^N$ we set

$$\alpha + \beta := (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_N + \beta_N), \tag{6.15}$$

and we write $\beta \leqslant \alpha$ if $\beta_i \leqslant \alpha_i$ for every $i = 1, \ldots, N$. Note that, if $\beta \leqslant \alpha$ then $|\beta| \leqslant |\alpha|$ while the converse does not hold (consider $\alpha = (3, 0)$ and $\beta = (2, 2)$). Note that the monomial notation allows writing expressions that appear as a product of vectors. What we mean is that an expression like $x^\alpha y^\beta$ makes sense because it is nothing but the product of two monomials:

$$x^\alpha y^\beta = (x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \cdots \cdot x_N^{\alpha_N}) \cdot (y_1^{\beta_1} \cdot x_2^{\beta_2} \cdot \cdots \cdot x_N^{\beta_N}). \tag{6.16}$$

In particular, we have $x^\alpha y^\alpha = (x_1 y_1, \ldots, x_N y_N)^\alpha$ and $x^\alpha x^\beta = x^{\alpha+\beta}$.

If $\beta \leqslant \alpha$, and only in this case, we set

$$\alpha - \beta := (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots, \alpha_N - \beta_N). \tag{6.17}$$

Also, we define the **factorial** $\alpha! := \alpha_1! \alpha_2! \ldots \alpha_N!$ and, when $\beta \leqslant \alpha$, the **binomial coefficient**

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_N}{\beta_N} = \frac{\alpha!}{\beta!(\alpha - \beta)!} = \frac{\alpha_1!}{\beta_1!(\alpha_1 - \beta_1)} \cdot \frac{\alpha_2!}{\beta_2!(\alpha_2 - \beta_2)} \cdots \frac{\alpha_N!}{\beta_N!(\alpha_N - \beta_N)}. \tag{6.18}$$

After that, for $\alpha \in \mathbb{N}^N$, $m \in \mathbb{N}$ and arbitrary $x, y \in \mathbb{R}^N$ we can write the Newton multinomial formulas in the following concise forms

$$(x + y)^\alpha = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} x^\beta \, y^{\alpha - \beta} = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} y^\beta \, x^{\alpha - \beta}, \tag{6.19}$$

$$(x_1 + x_2 + \cdots + x_N)^m = \sum_{|\beta| = m} \frac{m!}{\beta!} x^\beta. \tag{6.20}$$

The expansion (6.19) is known as multi-binomial theorem. For the sake of clarity, let us show how (6.19) is obtained. By the binomial theorem, we have

$$\begin{aligned}(x+y)^{\alpha} &= (x_1+y_1,\dots,x_N+y_N)^{\alpha}\\ &= (x_1+y_1)^{\alpha_1}\cdots(x_N+y_N)^{\alpha_1}\\ &= \left(\sum_{\beta_1=0}^{\alpha_1}\binom{\alpha_1}{\beta_1}x_1^{\beta_1}y_1^{\alpha_1-\beta_1}\right)\cdots\left(\sum_{\beta_N=0}^{\alpha_N}\binom{\alpha_N}{\beta_N}x_1^{\beta_N}y_1^{\alpha_N-\beta_N}\right)\\ &= \sum_{\beta_1=0}^{\alpha_1}\cdots\sum_{\beta_N=0}^{\alpha_N}\left[\binom{\alpha_1}{\beta_1}x_1^{\beta_1}y_1^{\alpha_1-\beta_1}\cdots\binom{\alpha_N}{\beta_N}x_1^{\beta_N}y_1^{\alpha_N-\beta_N}\right]\\ &= \sum_{\beta\leqslant\alpha}\left[\binom{\alpha}{\beta}x^{\beta}y^{\alpha-\beta}\right].\end{aligned}$$

Finally, the $\alpha$-**order partial derivative** symbol $D^{\alpha}$ is defined by

$$\partial^{\alpha} = D^{\alpha} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_N^{\alpha_N}} = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_N^{\alpha_N} = (\partial_1,\dots,\partial_N)^{(\alpha_1,\dots,\alpha_N)} \tag{6.21}$$

where the notation $\partial_i$ as a shortcut for $\frac{\partial}{\partial x_i}$.

**6.34. Remark.** Formally, the multi-index notation does not permit to express all possible *mixed* partial derivatives. Indeed, it is not possible to express the second-order mixed partial derivative $\partial_2\partial_1 = \partial^2/(\partial x_2\partial x_1)$, but only the mixed partial derivative $\partial_1\partial_2 = \partial^2/(\partial x_1\partial x_2)$ which is associated with the $2$-dimensional multi-index $(1,1)$ of absolute value $|(1,1)| = 2$. However, this does not cause any trouble because, in the theory of distributions, the Schwarz theorem on the symmetry of mixed partial derivatives always holds. *Mixed partial derivatives do not depend on the order of differentiation with respect to the different variables.*

### 6.2.2. The vector spaces $C^k(\Omega)$ ($k\in\bar{\mathbb{N}}, \Omega\subseteq\mathbb{R}^N, \Omega$ open)

**6.35. Definition.** We say that a function $f$ defined in the nonempty open set $\Omega\subseteq\mathbb{R}^N$ is of class $C^k$ **on** $\Omega$, $k\in\mathbb{N}$, if $D^{\alpha}f$ exists and is continuous for every multi-index $\alpha\in\mathbb{N}^N$ such that $|\alpha|\leqslant k$. We say that $f$ is of class $C^{\infty}$ (or infinitely differentiable) **on** $\Omega$, if it is of class $C^k$ for every $k\in\mathbb{N}$.

Let us recall that for a function $f$ of class $C^{k+1}$ on $\Omega$ the Taylor fomula holds:

$$f(x+y) = \sum_{|\alpha|\leqslant k}\frac{y^{\alpha}}{\alpha!}D^{\alpha}f(x) + R_{k+1}(x,y) \tag{6.22}$$

with

$$R_{k+1}(x,y) := (k+1)\sum_{|\alpha|=k+1}\frac{y^{\alpha}}{\alpha!}\int_0^1(1-s)^kD^{\alpha}f(x+sy)\,\mathrm{d}s \tag{6.23}$$

for every couple of points $x,y\in\mathbb{R}^N$ such that the **closed** segment $[x,x+y]$ is included in $\Omega$. Recall that the **closed** segment $[x,x+y]$ is given by $\{x+ty :: 0\leqslant t\leqslant 1\}$.

Given two functions $f,g$ of class $C^k$ on $\Omega$, the product function $fg$ is still of class $C^k$ on $\Omega$ and the Leibniz formula holds:

$$D^{\alpha}(fg) = \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}D^{\beta}f\,D^{\alpha-\beta}g \quad \textbf{for any } \alpha\in\mathbb{N}^N :: |\alpha|\leqslant k. \tag{6.24}$$

**6.36. Definition.** Let $\Omega$ be a nonempty open set of $\mathbb{R}^N$. We denote by $C^k(\Omega, \mathbb{K})$ the set of all $\mathbb{K}$-valued functions defined on $\Omega$ and of class $C^k$ on $\Omega$. As usual, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and often, to shorten notation, we will simply use the symbol $C^k(\Omega)$.

A unital or unitary module is a module over a unital ring in which the identity element of the ring acts as the identity on the module. A unital algebra $A$ over a field $\mathbb{K}$ is an algebra over $\mathbb{K}$ which is unital as a ring

**6.37. Proposition.** *The set $C^k(\Omega)$, when endowed with the usual operations of addition, multiplication by a scalar, and multiplication of functions, becomes an* **unital** *algebra over $\mathbb{K}$, the multiplicative identity being the characteristic function $x \in \Omega \mapsto 1 \in \mathbb{K}$ of $\Omega$.*

**PROOF.** It is straightforward to check that $C^k(\Omega)$ is a vector space. That $C^k(\Omega)$ is closed under the multiplication operation $(f, g) \in C^k(\Omega) \times C^k(\Omega) \mapsto fg$ follows from the Leibniz rule (6.24) and the fact that the product of continuous function is continuous. ■

### 6.2.3. The (Hausdorff) separated locally convex space $\mathcal{E}^k(\Omega)$ ($k \in \bar{\mathbb{N}}, \Omega \subseteq \mathbb{R}^N, \Omega$ open)

Let $\Omega$ be a nonempty open set of $\mathbb{R}^N$. Let $K \in \mathfrak{K}_\Omega$ be a compact subset of $\Omega$ and let $k \in \bar{\mathbb{N}}$. For any $f \in C^k(\Omega)$ and any $m \in \bar{\mathbb{N}}$ such that $m \leqslant k$ we set

$$\mathfrak{p}_{K,m}(f) := \sup_{|\alpha| \leqslant m} \sup_{x \in K} |\mathsf{D}^\alpha f(x)|. \tag{6.25}$$

Of course, when $k = \infty$ the condition $(m \leqslant k)$ is unnecessary.

Note that, according to WEIERSTRASS's extreme value theorem, the suprema in (6.25) can be replaced by the maxima.

**Example 6.38.** It can be useful to make explicit the expression of $\mathfrak{p}_{K,m}(f)$ for some values of $m \in \mathbb{N}$. For $m = 0$ we have $|\alpha| = 0$ if, and only if, $\alpha = (0, \ldots, 0) \in \mathbb{N}^N$ and therefore

$$\mathfrak{p}_{K,0}(f) = \sup_{x \in K} |f(x)| = \max_{x \in K} |f(x)|.$$

For $N = 3$ and $|\alpha| = 1$ we have $\{|\alpha| \leqslant 1\} = \{|\alpha| = 0\} \cup \{|\alpha| = 1\} = \{(0,0,0)\} \cup \{(1,0,0), (0,1,0), (0,0,1)\}$ and therefore

$$\begin{aligned}\mathfrak{p}_{K,1}(f) &= \mathfrak{p}_{K,0}(f) \vee \left(\sup_{i \in \mathbb{N}_3} \sup_{x \in K} |\partial_i f(x)|\right) \\ &= \left(\max_{x \in K} |f(x)|\right) \vee \left(\max_{i \in \mathbb{N}_3} \max_{x \in K} |\partial_i f(x)|\right).\end{aligned}$$

For $N = 3$ and $|\alpha| = 2$ we have $\{|\alpha| \leqslant 2\} = \{|\alpha| = 0\} \cup \{|\alpha| = 1\} \cup \{|\alpha| = 2\} = \{(0,0,0)\} \cup \{(1,0,0), (0,1,0), (0,0,1)\} \cup \{(2,0,0), (0,2,0), (0,0,2), (0,1,1), (1,0,1), (1,1,0)\}$, and therefore

$$\mathfrak{p}_{K,2}(f) = \mathfrak{p}_{K,1}(f) \vee \left(\sup_{(i,j) \in \mathbb{N}_3^2} \sup_{x \in K} |\partial^2_{ij} f(x)|\right).$$

Eventually, since for every $a \in \Omega$ the singleton $\{a\} \in \mathfrak{K}_\Omega$, we have

$$\mathfrak{p}_{\{a\},0}(f) = |f(a)| \quad \textbf{and} \quad \mathfrak{p}_{\{a\},m}(f) = \sup_{|\alpha| \leqslant m} |\mathsf{D}^\alpha f(a)|.$$

The following result holds.

**6.39. Proposition.** *When the compact set $K$ varies over all possible compact subsets of $\Omega$, and $m$ varies over all natural number less than or equal to $k$, the family $(\mathfrak{p}_{K,m})_{K \in \mathfrak{K}_\Omega, m \leqslant k}$ describes a filtering and total family of seminorms on $C^k(\Omega)$.*

**PROOF.** Every $\mathfrak{p}_{K,m}$ is a seminorm because for every multi-index $|\alpha| \leqslant m$ the functional $\mathfrak{p}_{K,\alpha}(f) := \sup_{x\in K} |\mathrm{D}^\alpha f(x)|$ is a seminorm on $C^k(\Omega)$ and $\mathfrak{p}_{K,m}$ is nothing but the **superior envelope** of the finite number of seminorms $(\mathfrak{p}_{K,\alpha})_{|\alpha|\leqslant m}$. The family $(\mathfrak{p}_{K,m})_{K\in\mathfrak{K}_\Omega, m\leqslant k}$ is filtering because if $K = K_1 \cup K_2$ and $m = m_1 \vee m_2$ then

$$\mathfrak{p}_{K_1,m_1} \vee \mathfrak{p}_{K_2,m_2} \leqslant \mathfrak{p}_{K,m}.$$

The family $(\mathfrak{p}_{K,m})_{K\in\mathfrak{K}_\Omega, m\leqslant k}$ separates the points (is total) because, $\mathfrak{p}_{\{a\},0}(f) = |f(a)|$ for every $a \in \Omega$. ■

**6.40. Definition.** The natural topology of $C^k(\Omega)$ is, by definition, the locally convex topology $\tau_\Omega^k$ induced by the filtering and separating family of seminorms $(\mathfrak{p}_{K,m})$ defined by (6.25). We denote by $\boldsymbol{\mathcal{E}}^k(\Omega)$ the locally convex space $(C^k(\Omega), \tau_\Omega^k)$.

**6.41. Remark.** Note that, for $k = 0$, we have $\boldsymbol{\mathcal{E}}^0(\Omega) = \boldsymbol{\mathfrak{C}}(\Omega)$. We introduced the space $\boldsymbol{\mathfrak{C}}(\Omega)$ separately because for $\boldsymbol{\mathfrak{C}}(\Omega)$ no differentiable structure is needed on the topological space $\Omega$. Indeed, we introduced $\boldsymbol{\mathfrak{C}}(\Omega)$ for any topological space $\Omega$ which is a $\sigma$-locally compact Hausdorff space. Here, instead, we restrict ourselves to open subsets of $\mathbb{R}^N$.

**6.42. Corollary.** *The space $\boldsymbol{\mathcal{E}}^k(\Omega)$ is a (Hausdorff) separated locally convex space.*

**PROOF.** It follows immediately from **Proposition 6.39** and **Proposition 4.15**. ■

### 6.2.4. Convergence in $\boldsymbol{\mathcal{E}}^k(\Omega)$

If one wants to give a name to the topology $\tau_\Omega^k$, we recall that, for every $k \in \bar{\mathbb{N}}$, it is sometimes referred to as the **$k$-smooth uniform convergence on all compact subsets** of $\Omega$. When $k = \infty$, it is simply referred to as the **smooth uniform convergence on all compact subsets** of $\Omega$. This stems from the characterization of $\tau_\Omega^k$ stated in the next result.

**6.43. Proposition.** *A generalized sequence $(f_\lambda)_{\lambda\in\Lambda}$ of elements of $C^k(\Omega)$ converges to $f \in C^k(\Omega)$ in $\boldsymbol{\mathcal{E}}^k(\Omega)$ if, and only if, for every multi-index $\alpha \in \mathbb{N}^N$ with $|\alpha| \leqslant k$, and for every compact subset $K \subseteq \Omega$, the generalized sequence of functions $(\mathrm{D}^\alpha f_\lambda)_{\lambda\in\Lambda}$ uniformly converges to $\mathrm{D}^\alpha f$ on $K$.*

**PROOF.** As a consequence of **Proposition 4.34**, the convergence of $(f_\lambda)_{\lambda\in\Lambda}$ to $f$, in $\boldsymbol{\mathcal{E}}^k(\Omega)$, is equivalent to the condition $\lim_\Lambda \mathfrak{p}_{K,m}(f_\lambda - f) = 0$, for every $m \leqslant k$ and every compact subset $K \in \mathfrak{K}_\Omega$. ■

**6.44. Remark.** Note that, when $k < \infty$, it is possible to obtain the same topology by considering the "*simpler*" family of seminorms

$$f \in C^k(\Omega) \mapsto \mathfrak{p}_{K,k}(f) := \sup_{|\alpha|\leqslant k} \sup_{x\in K} |\mathrm{D}^\alpha f(x)|,$$

in which just the compact set $K$ varies in $\mathfrak{K}_\Omega$, that is, the family $(\mathfrak{p}_{K,k})_{K\in\mathfrak{K}_\Omega}$. On the other hand, when $k = \infty$, the relations

$$\lim_{n\to\infty} \left( \sup_{\alpha\in\mathbb{N}^N} \sup_{x\in K} |\mathrm{D}^\alpha (f_n(x) - f(x))| \right) = 0$$

and

$$\lim_{n\to\infty} \left( \sup_{|\alpha|\leqslant m} \sup_{x\in K} |\mathrm{D}^\alpha (f_n(x) - f(x))| \right) = 0 \quad \forall m \in \mathbb{N}$$

have different meanings. The more "complicate" family of seminorm $(\mathfrak{p}_{K,m})_{K\in\mathfrak{K}_\Omega, m\leqslant k}$ allows unifying the treatment of the cases $k<\infty$ and $k=\infty$.

### 6.2.5. $\boldsymbol{\mathcal{E}}^k(\Omega)$ admits a countable basis of continuous seminorms

The following result holds.

**6.45. Proposition.** *The space $\boldsymbol{\mathcal{E}}^k(\Omega)$ admits a* **countable** *basis of continuous seminorms. Therefore, $\boldsymbol{\mathcal{E}}^k(\Omega)$ is a pre-Fréchet space.*

**PROOF.** It is sufficient to take, as a basis of continuous seminorms, the family

$$(\mathfrak{p}_{K_j,m})_{j\in\mathbb{N}, m\leqslant k}$$

associated with the countable family of domains $(K_j := \bar{\Omega}_j)_{j\in\mathbb{N}}$ where $(\Omega_j)_{j\in\mathbb{N}}$ is an exhaustion of $\Omega$ by open and relatively compact sets (cf. **Remark ?**). Indeed, for any $K\in\mathfrak{K}_\Omega$ there exists $j_\star\in\mathbb{N}$ such that $K\subseteq K_j$ for every $j\geqslant j_\star$, and therefore $\mathfrak{p}_{K,m}\preccurlyeq\mathfrak{p}_{K_{j_\star},m}$. But then, the assertion follows by **Corollary 4.31**. ■■■■

### 6.2.6. Completeness of $\boldsymbol{\mathcal{E}}^k(\Omega)$

**6.46. Theorem.** *For every $k\in\bar{\mathbb{N}}$, the space $\boldsymbol{\mathcal{E}}^k(\Omega)$ is a Frechét space.*

**6.47. Remark.** When $k=0$, the proof that we give works also for $\Omega$ a $\sigma$-locally compact Hausdorff space.

**PROOF.** According to **Proposition 6.45**, the (Hausdorff) separated locally convex space $\boldsymbol{\mathcal{E}}^k(\Omega)$ admits *a countable basis of continuous seminorms*. Therefore, it is sufficient to prove that $\boldsymbol{\mathcal{E}}^k(\Omega)$ is (sequentially) complete. We split the proof into three steps.

Recall that since $\boldsymbol{\mathcal{E}}^k(\Omega)$ is a first countable topological vector space, the notion of complete space reduces to that of sequentially complete space (cf. **Remark 4.44**).

**Step 1 (The space $\boldsymbol{\mathcal{E}}^0(\Omega)=\boldsymbol{\mathcal{C}}(\Omega)$ is complete).** Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\boldsymbol{\mathcal{C}}(\Omega)$. For every $x\in\Omega$, the numerical sequence $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the topology of $\mathbb{K}$. Since $\mathbb{K}$ is complete, there exists a number $f(x)\in\mathbb{K}$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \textbf{in } \mathbb{K}.$$

Thus, it is well-defined the function $x\in\Omega\mapsto f(x)\in\mathbb{K}$. Next, let $K\in\mathfrak{K}_\Omega$ be a compact subset of $\Omega$, and denote by $g_n := f_{n|K}$ the restriction of $f_n$ to $K$. By assumption, the sequence $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space $\boldsymbol{\mathcal{C}}(K)$ and, therefore, $(g_n)_{n\in\mathbb{N}}$ converges uniformly on $K$ to some function $g_K\in\boldsymbol{\mathcal{C}}(K)$. Clearly, $f_{|K}\equiv g_K$ because uniform convergence and pointwise convergence are compatible. By the arbitrariness of $K\in\mathfrak{K}_\Omega$, the sequence $(f_n)_{n\in\mathbb{N}}$ uniformly converges to $f$ on every compact set $K\in\mathfrak{K}_\Omega$, and this is precisely the convergence of $(f_n)_{n\in\mathbb{N}}$ to $f$ in $\boldsymbol{\mathcal{C}}(\Omega)$.

Do not confuse $\boldsymbol{\mathcal{C}}(K)$ and $\boldsymbol{\mathcal{K}}_K(\Omega)$.

**Step 2 (the graph of $\nabla$ is closed).** We observe that if $\Omega$ is an open subset of $\mathbb{R}^N$, the graph of the gradient operator

$$\nabla\colon f\in\boldsymbol{\mathcal{E}}^1(\Omega)\mapsto\nabla f\in\boldsymbol{\mathcal{E}}^0(\Omega)^N$$

is closed in $\boldsymbol{\mathcal{E}}^0(\Omega)\times\boldsymbol{\mathcal{E}}^0(\Omega)^N$. In other words, if $(f_n,\nabla f_n)_{n\in\mathbb{N}}$ is a sequence in $\boldsymbol{\mathcal{E}}^1(\Omega)\times(\boldsymbol{\mathcal{E}}^0(\Omega))^N$ converging in $\boldsymbol{\mathcal{E}}^0(\Omega)\times(\boldsymbol{\mathcal{E}}^0(\Omega))^N$ to some $(f,\boldsymbol{g})\in\boldsymbol{\mathcal{E}}^0(\Omega)\times(\boldsymbol{\mathcal{E}}^0(\Omega))^N$ then $(f,\boldsymbol{g})$ is still a point of the graph of $\nabla$, i.e., necessarily $\boldsymbol{g}=\nabla f$. The fact that the graph of $\nabla$ is closed is a consequence of the following well-known result: **If** *$(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}^1(\Omega)$ such that $(\nabla f_n)_{n\in\mathbb{N}}$ converges in $\mathcal{E}^0(\Omega)^N$ to some function $\boldsymbol{g}\in\mathcal{E}^0(\Omega)^N$, and if in every connected component of $\Omega$ there exists a point $x$ such that the numerical sequence $(f_n(x))_{n\in\mathbb{N}}$ converges in $\mathbb{K}$,* **then** *there exists $f\in\mathcal{E}^1(\Omega)$ such that $(f_n)_{n\in\mathbb{N}}\to f$ in $\mathcal{E}^1(\Omega)$ and $\boldsymbol{g}=\nabla f$.*

**Step 3 ($\mathcal{E}^k(\Omega)$ is (sequentially) complete for any $k\in\bar{\mathbb{N}}$).** Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{E}^k(\Omega)$. For every $\alpha\in\mathbb{N}^N$ such that $|\alpha|\leqslant k$, the sequence $(\mathsf{D}^\alpha f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}^0(\Omega)$. Since $\mathcal{E}^0(\Omega)$ is complete, for every $\alpha\in\mathbb{N}^N$ there exists $\boldsymbol{g}^{|\alpha|}\in\mathcal{E}^0(\Omega)^{N^{|\alpha|}}$ such that

$$\nabla^{|\alpha|}f_n\to\boldsymbol{g}^{|\alpha|}\quad\textbf{in } \mathcal{E}^0(\Omega).$$

We set $f:=\boldsymbol{g}^0$, $\boldsymbol{g}:=\boldsymbol{g}^{|1|}$, and we argue by induction using the fact that the gradient has a closed graph. Precisely

$$\begin{array}{rcl} f_n & \to & f \\ \nabla f_n & \to & \boldsymbol{g} \end{array} \;\Rightarrow\; \boldsymbol{g}=\nabla f$$

$$\begin{array}{rcl} \nabla f_n & \to & \nabla f \\ \nabla^2 f_n & \to & \boldsymbol{g}^2 \end{array} \;\Rightarrow\; \boldsymbol{g}^2=\nabla\nabla f=\nabla^2 f$$

$$\vdots\quad\vdots\quad\vdots$$

$$\begin{array}{rcl} \nabla^{k-1} f_n & \to & \nabla^{k-1} f \\ \nabla^k f_n & \to & \boldsymbol{g}^k \end{array} \;\Rightarrow\; \boldsymbol{g}^k=\nabla\nabla^{k-1}f=\nabla^k f.$$

But then, for every multi-index $\alpha\in\mathbb{N}^N$ such that $|\alpha|\leqslant k$, the sequence $(\mathsf{D}^\alpha f_n)_{n\in\mathbb{N}}$ converges in $\mathcal{E}^0(\Omega)$ to $\mathsf{D}^\alpha f$. In other words, $(f_n)_{n\in\mathbb{N}}\to f$ in $\mathcal{E}^k(\Omega)$. ■■■■

### 6.2.7. Appendix: termwise differentiation of sequences

**Example 6.48.** Consider the sequence of real-valued functions defined by $f_n(x):=\frac{nx}{1+n^2x^2}$ for every $n\in\mathbb{N}$ and for all $x\in\mathbb{R}$. Let $f\equiv 0$ be the function identically equal to zero. Clearly, one has $f_n\to f$ pointwise in $\mathbb{R}$, but the convergence is not uniform because

$$f_n(1/n)=\frac{1}{2}\quad\forall n\in\mathbb{N}. \tag{6.26}$$

Indeed, recall that if a sequence of real-valued function $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f$ in $\mathbb{R}$, then $f_n(x_n)-f(x_n)\to 0$ for any sequence $(x_n)_{n\in\mathbb{N}}$ of points of $\mathbb{R}$. This is because of

$$|f_n(x_n)-f(x_n)|<\sup_{x\in\mathbb{R}}|f_n(x)-f(x)|\xrightarrow{n\to+\infty}0.$$

Note that $(f_n)_{n\in\mathbb{N}}$ gives an example of a sequence in $\mathcal{E}^0(\mathbb{R})$ which converges pointwise to an element of $\mathcal{E}^0(\mathbb{R})$, but does not converge in $\mathcal{E}^0(\mathbb{R})$ because, for example, the convergence is not uniform in any compact set passing through the origin as (6.26) shows. Also, we have that $\partial_x f_n(x)=\frac{(n-n^3x^2)}{(1+n^2x^2)^2}$. Hence, $\partial_x f_n(x)\to 0=\partial_x f(x)$ if $x\neq 0$, but $\partial_x f_n(0)=n\to+\infty\neq 0=\partial_x f(0)$.

Example 6.48 shows the existence of a sequence $(f_n,\partial_x f_n)_{n\in\mathbb{N}}$ in the graph of the gradient operator $\partial_x\colon f\in\mathcal{E}^1(\mathbb{R})\mapsto\partial_x f\in\mathcal{E}^0(\mathbb{R})$ that does not converge to an element of the graph of $\partial_x$. This is because $(f_n,\partial_x f_n)_{n\in\mathbb{N}}$ does not converge in $\mathcal{E}^0(\Omega)\times\mathcal{E}^0(\Omega)$. Indeed, the aim of this section is to show that the graph of the gradient operator

$$\nabla\colon f\in\mathcal{E}^1(\Omega)\mapsto\nabla f\in\mathcal{E}^0(\Omega)^N$$

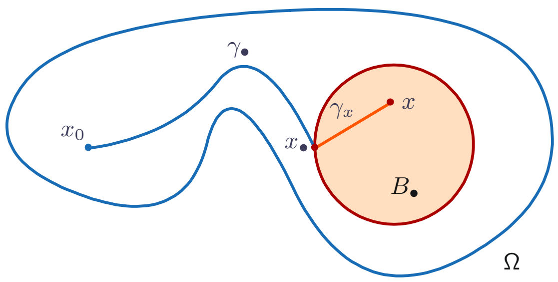



**Figure 6.5.** Let $B_\bullet\in\mathfrak{K}_\Omega$ be a closed ball in $\Omega$ with nonempty interior $B_\circ$ (i.e., that contains more than one point), and let $\delta_\bullet$ be the distance of $B_\bullet$ to $x_0\in\Omega$. It is well-known that this distance is achieved, i.e., that there exists $x_\bullet\in B_\bullet$ such that $\delta_\bullet=|x_\bullet-x_0|$. We denote by $\gamma_\bullet$ a smooth curve in $\Omega$ joining $\gamma_\bullet(0)=x_0$ and $\gamma_\bullet(1)=x_\bullet$, and by $\Gamma_\bullet$ its image in $\Omega$.

is closed in $\boldsymbol{\mathcal{E}}^0(\Omega)\times\boldsymbol{\mathcal{E}}^0(\Omega)^N$. In other words, we want to show that if $(f_n,\nabla f_n)_{n\in\mathbb{N}}$ is a sequence in $\boldsymbol{\mathcal{E}}^1(\Omega)\times(\boldsymbol{\mathcal{E}}^0(\Omega))^N$ converging in $\boldsymbol{\mathcal{E}}^0(\Omega)\times(\boldsymbol{\mathcal{E}}^0(\Omega))^N$ to some $(f,\boldsymbol{g})\in\boldsymbol{\mathcal{E}}^1(\Omega)\times(\boldsymbol{\mathcal{E}}^0(\Omega))^N$ then $(f,\boldsymbol{g})$ is still a point of the graph of $\nabla$ (i.e., necessarily $f\in\boldsymbol{\mathcal{E}}^1(\Omega)$ and $\boldsymbol{g}=\nabla f$). In fact, we prove something stronger.

**6.49. Theorem.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{\mathcal{E}}^1(\Omega)$. If $(\nabla f_n)_{n\in\mathbb{N}}$ converges in $\boldsymbol{\mathcal{E}}^0(\Omega)^N$ to some function $\boldsymbol{g}\in\boldsymbol{\mathcal{E}}^0(\Omega)^N$, and if in every connected component of $\Omega$ there exists a point $x_0$ such that the numerical sequence $(f_n(x_0))_{n\in\mathbb{N}}$ converges in $\mathbb{K}$, then there exists $f\in\boldsymbol{\mathcal{E}}^1(\Omega)$ such that $(f_n)_{n\in\mathbb{N}}\to f$ in $\boldsymbol{\mathcal{E}}^1(\Omega)$ and $\boldsymbol{g}=\nabla f$.*

**PROOF.** We can assume that $\Omega$ is connected. Let $B_\bullet\in\mathfrak{K}_\Omega$ be a closed ball in $\Omega$ with nonempty interior $B_\circ$ (i.e., that contains more than one point), and let $\delta_\bullet$ be the distance of $B_\bullet$ to $x_0\in\Omega$. It is well-known that this distance is achieved, i.e., that there exists $x_\bullet\in B_\bullet$ such that $\delta_\bullet=|x_\bullet-x_0|$. We denote by $\gamma_\bullet$ a smooth curve in $\Omega$ joining $\gamma_\bullet(0)=x_0$ and $\gamma_\bullet(1)=x_\bullet$, and by $\Gamma_\bullet$ its image in $\Omega$ (cf. Figure 6.5).

Next, observe that for every $m,n\in\mathbb{N}$ the function

$$\varphi_{m,n}:=f_m-f_n$$

is continuously differentiable on the compact set $K:=B_\bullet\cup\Gamma_\bullet$ because $(f_n)_{n\in\mathbb{N}}$ is in $\boldsymbol{\mathcal{E}}^1(\Omega)^N$. Hence, if we denote by $\gamma_x$ the line segment joining $\gamma_x(0)=x_\bullet$ and $\gamma_x(1)=x$, then for every $x\in B_\bullet$ we have

$$\begin{aligned}
|\varphi_{m,n}(x)-\varphi_{m,n}(x_0)| &\leqslant |(\varphi_{m,n}\circ\gamma_x)(1)-(\varphi_{m,n}\circ\gamma_x)(0)|+|(\varphi_{m,n}\circ\gamma_\bullet)(1)-(\varphi_{m,n}\circ\gamma_\bullet)(0)|\\
&\leqslant \int_0^1|\partial_t(\varphi_{m,n}\circ\gamma_x)(t)|\,\mathrm{d}t+\int_0^1|\partial_t(\varphi_{m,n}\circ\gamma_\bullet)(t)|\,\mathrm{d}t\\
&= \int_0^1|\nabla\varphi_{m,n}(\gamma_x(t))|\,|\gamma_x'(t)|\,\mathrm{d}t+\int_0^1|\nabla\varphi_{m,n}(\gamma_\bullet(t))|\,|\gamma_\bullet'(t)|\,\mathrm{d}t\\
&\leqslant (L(\gamma_x)+L(\gamma_\bullet))\sup_{y\in K}|\nabla\varphi_{m,n}(y)|,
\end{aligned}$$

where we denoted by $L(\gamma)$ the length of $\gamma$. Clearly, since $\gamma_x$ is a line segment contained in $B_\bullet$ we have $L(\gamma_\bullet)\leqslant\operatorname{diam}(B_\bullet)$ and, therefore,

$$|\varphi_{m,n}(x)-\varphi_{m,n}(x_0)|\leqslant(\operatorname{diam}(B_\bullet)+L(\gamma_\bullet))\sup_{y\in K}|\nabla\varphi_{m,n}(y)|.$$

In particular, by the reverse triangular inequality, we get

$$\sup_{x\in K}|\varphi_{m,n}(x)|\leqslant|\varphi_{m,n}(x_0)|+(\operatorname{diam}(B_\bullet)+L(\gamma_\bullet))\sup_{x\in K}|\nabla\varphi_{m,n}(x)|.$$

On the other hand, by hypotheses, we know that $\sup_{x\in K}|\nabla\varphi_{m,n}(x)|\to 0$ for $(m,n)\to\infty$ because $\nabla f_n$ converges uniformly to $\boldsymbol{g}$ on every compact subset of $\Omega$ and, therefore, it is uniformly Cauchy in $\boldsymbol{\mathcal{E}}^0(K)$. Also $|\varphi_{m,n}(x_0)|\to 0$ because, by hypothesis, $(f_n(x_0))_{n\in\mathbb{N}}$ converges in $\mathbb{K}$. It follows that

$$\lim_{(m,n)\to\infty}\sup_{x\in K}|\varphi_{m,n}(x)|=0.$$

But since $\varphi_{m,n}:=f_m-f_n$, this means that $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy in $K$ and, therefore, since $\boldsymbol{\mathcal{E}}^0(K)$ is a complete space, there exists $f\in\boldsymbol{\mathcal{E}}^0(K)$ such that

$$(f_n)_{n\in\mathbb{N}}\to f\quad\text{uniformly in } K.$$

Note that such a limit $f$ depends only on the sequence $(f_n)_{n\in\mathbb{N}}$ and not on $K$ because, a posteriori, $f$ is uniquely determined as the pointwise limit in $K$ of $(f_n)_{n\in\mathbb{N}}$.

In particular, we have $(f_n)_{n\in\mathbb{N}}\to f$ uniformly in $B_\bullet$. This entails that for every $\varphi\in C_c^\infty(B_\circ)$, $B_\circ=(B_\bullet)^\circ$, it is well defined the functional (distribution)

$$\langle f_n,\varphi\rangle:=\int_{B_\circ}f_n(x)\varphi(x)\,\mathrm{d}x,$$

and $\langle f_n,\varphi\rangle\to\langle f,\varphi\rangle$ when $n\to\infty$. Also, for every $\boldsymbol{\varphi}\in C_c^\infty(B_\circ,\mathbb{R}^N)$ it is well defined the functional (derivative of a distribution)

$$\langle\nabla f_n,\boldsymbol{\varphi}\rangle:=-\int_{B_\circ}f_n(x)\,\mathrm{div}\boldsymbol{\varphi}(x)\,\mathrm{d}x,$$

and, moreover, $\langle\nabla f_n,\boldsymbol{\varphi}\rangle\to\langle\nabla f,\boldsymbol{\varphi}\rangle$ when $n\to\infty$. On the other hand, by uniform continuity of $\nabla f_n$ to $\boldsymbol{g}$ in $B_\bullet$, we know that

$$\langle\nabla f_n,\boldsymbol{\varphi}\rangle\to\langle\boldsymbol{g},\boldsymbol{\varphi}\rangle\quad\forall\boldsymbol{\varphi}\in C_c^\infty(B_\circ).$$

It follows that $\nabla f\equiv\boldsymbol{g}$ in $B_\circ$ because of the fundamental theorem of the calculus of variations (which assures that the limit in the sense of the distributions is unique). In particular, $f\in\boldsymbol{\mathcal{E}}^1(B_\circ)$.

Overall, in the statement's hypothesis, we know that for every closed ball $B_\bullet\in\mathfrak{K}_\Omega$ the gradient $\nabla f$ of $f$ exists in a classical sense in $B_\circ$ and $\nabla f=\boldsymbol{g}$ in $B_\circ$. In particular, $\nabla f$ is continuous on $B_\circ$.

By the arbitrarity of $B_\bullet$ we conclude that there exists a continuous function $f$ defined in the whole of $\Omega$ such that $f_n\to f$ in $\boldsymbol{\mathcal{E}}^1(B_\circ)$ for every open ball $B_\circ$ well inside of $\Omega$ (i.e., such that $B_\bullet=\overline{B_\circ}\subseteq\Omega$) and, moreover, there holds that $\nabla f=\boldsymbol{g}$ in $\Omega$. But then, if $K$ is an arbitrary compact subset of $\Omega$, one can cover $K$ by a finite number of open balls that are well inside of $\Omega$ and, therefore, to conclude the proof, it is sufficient to observe that if a sequence converges uniformly on two subsets of $\Omega$ then it also converges uniformly on their union. ■■■■

**6.50. Remark.** Theorem 6.49 is sufficient to prove the completeness of the spaces $\boldsymbol{\mathcal{E}}^k(\Omega)$ and, therefore, we are happy with that. However, for the sake of completeness, we recall that by using $\varepsilon,\delta$ techniques, one can prove a one-dimensional analog of it, which does not assume continuity of the derivatives. Precisely, the following result holds (see, e.g., RUDIN, WALTER. *Principles of mathematical analysis*. New York: McGraw-Hill, 1964). *Let $K$ be a compact interval in $\mathbb{R}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued functions that are differentiable on $K$. Suppose that $(f_n')_{n\in\mathbb{N}}$ converges uniformly on $K$ to some function $g$ and that, for some $x_0\in K$, the numerical sequence $(f_n(x_0))_{n\in\mathbb{N}}$ converges. Then there exists a function $f$ that is differentiable on $K$, such that $(f_n)_{n\in\mathbb{N}}\to f$ uniformly on $K$ and $f'(x)=g(x)=\lim_{n\to+\infty}f_n'(x)$ for all $x\in K$.*

## 6.3 The spaces $\mathfrak{D}^k(\Omega)$ $(k\in\bar{\mathbb{N}},\Omega\subseteq\mathbb{R}^N,\Omega$ open)

**Reminder. (Support of a continuous function, cf. Definition 6.15)** Let $f$ be a continuous function defined on the topological space $\Omega$. The **domain of nullity** of $f$, is the biggest open subset $U_\Omega(f)$ of $\Omega$ on which $f$ is identically zero. In other words, $U_\Omega(f):=\cup\{U\subseteq\Omega :: U \text{ is open in } \Omega \textbf{ and } f_{|U}\equiv 0\}$. The complement $\Omega\backslash U_\Omega(f)$ of the domain of nullity is, by definition, the **support** of $f$ and is denoted by $\operatorname{supp}_\Omega f$. Note that the support of $f$ coincides with the closure (in $\Omega$) of the set where $f$ is different from zero: $\operatorname{supp}_\Omega f=\overline{\{x\in\Omega :: f(x)\neq 0\}}$, the closure being taken in $\Omega$.

### 6.3.1. The vector space $C_c^k(\Omega)$ $(k\in\bar{\mathbb{N}},\Omega\subseteq\mathbb{R}^N,\Omega$ open)

**6.51. Definition.** Let $\Omega$ be a nonempty open set of $\mathbb{R}^N$. For any compact set $K\in\mathfrak{K}_\Omega$, and any $k\in\bar{\mathbb{N}}$, we denote by $C_K^k(\Omega)$ the subset of $C^k(\Omega)$ consisting of those functions whose support is contained in $K$. In other words:

$$C_K^k(\Omega):=\{f\in C^k(\Omega) :: \operatorname{supp}_\Omega f\subseteq K\}. \tag{6.27}$$

Then, we denote by $C_c^k(\Omega)$ the subset of $C^k(\Omega)$ consisting of those functions whose support is a compact subset of $\Omega$. In other words:

$$C_c^k(\Omega):=\{f\in C^k(\Omega) :: \operatorname{supp}_\Omega f\in\mathfrak{K}_\Omega\}. \tag{6.28}$$

Clearly, $C_K^k(\Omega)\subseteq C_c^k(\Omega)$ for every $K\in\mathfrak{K}_\Omega$, and $C_c^k(\Omega)\equiv\bigcup_{K\in\mathfrak{K}_\Omega}C_K^k(\Omega)$. Also, according to **Proposition ?**, there exists an increasing sequence $(\Omega_j)_{j\in\mathbb{N}}$, of open and relatively compact sets, covering $\Omega$ and such that $\bar{\Omega}_j\subseteq\Omega_{j+1}$ for every $j\in\mathbb{N}$. Clearly, if we set $K_j:=\bar{\Omega}_j$ we have

$$C_c^k(\Omega)\equiv\bigcup_{j\in\mathbb{N}}C_{K_j}^k(\Omega). \tag{6.29}$$

○

**6.52. Proposition.** *The set $C_c^k(\Omega)$, when endowed with the usual operations of addition, multiplication by a scalar, and multiplication of functions, becomes an algebra over $\mathbb{K}$ (**not** unital because $\chi_\Omega: x\in\Omega\mapsto 1\in\mathbb{K}$ is not in $C_c^k(\Omega)$). Moreover, $C_c^k(\Omega)$ is an **unital** module over the **unital** algebra $C^k(\Omega)$.*

A unital or unitary module is a module over a unital ring in which the identity element of the ring acts as the identity on the module. A unital algebra $A$ over a field $\mathbb{K}$ is an algebra over $\mathbb{K}$ which is unital as a ring

**PROOF.** Note that $C_c^k(\Omega)$ is a subalgebra of $C^k(\Omega)$. Indeed, for any $f,g\in C_c^k(\Omega)$ we have $fg\in C_c^k(\Omega)$ because of Leibniz rule (6.24), the fact that the product of continuous function is continuous, and the following relations concerning the support of a function: for any $f,g\in C^k(\Omega)$ (in particular for any $f,g\in C_c^k(\Omega)$) and any multi-index $|\alpha|\leqslant k$

$$\operatorname{supp}_\Omega(fg)\subseteq(\operatorname{supp}_\Omega f)\cap(\operatorname{supp}_\Omega g) \quad \textbf{and} \quad \operatorname{supp}_\Omega(\mathsf{D}^\alpha f)\subseteq\operatorname{supp}_\Omega f.$$

Finally, it is clear that the identity element of $C^k(\Omega)$, namely the characteristic function $\chi_\Omega$: $x\in\Omega\mapsto 1\in\mathbb{K}$, acts as the identity on the module. ■

We have already pointed out that when $\Omega$ is a general topological space, the space $C_c(\Omega)$ can reduce to the trivial vector space containing just the null function, and this is never the case when $\Omega$ is a locally compact Hausdorff space (due to **Urysohn Lemma ?**). But then, it is natural to ask under which assumptions on $\Omega\subseteq\mathbb{R}^N$ the vector space $C_c^k(\Omega)$ does not reduce to the null vector. It turns out that $C_c^k(\Omega)$ is **never** trivial because $C_c^\infty(\Omega)\subseteq C_c^k(\Omega)$ and the space $C_c^\infty(\Omega)$ is **never** trivial as the next result shows.

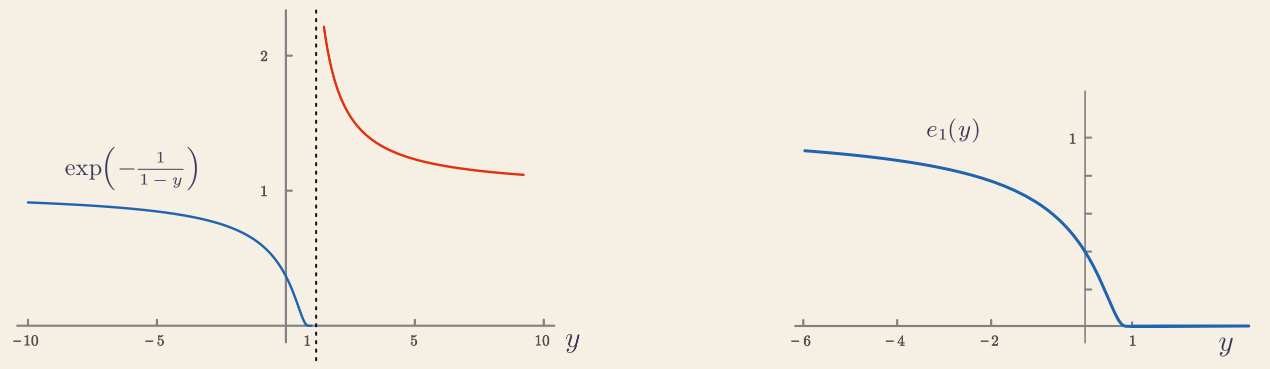



**Figure 6.6.** Left. The function $y\in\mathbb{R}\mapsto\exp(-1/(1-y))$. It is discontinuous at $1\in\mathbb{R}$. Right. The graph of the function $e_1\colon y\in\mathbb{R}\mapsto\exp(-1/(1-y))\chi_{(-\infty,1)}(y)$ used in the proof of **Proposition 6.53**.

**6.53. Proposition.** *The function* $\eta\colon\mathbb{R}^N\to\mathbb{K}$ *defined by*

$$\eta(x):=\exp\left(-\frac{1}{1-|x|^2}\right)\chi_B(x),\quad \textbf{that is}\quad \eta(x):=\begin{cases}\exp\left(-\frac{1}{1-|x|^2}\right) & \textbf{if } |x|<1,\\ 0 & \textbf{if } |x|\geqslant 1,\end{cases}$$

*where $B$ is the* **open** *unit ball of $\mathbb{R}^N$ and $\chi_B$ its characteristic function (here the convention is that $0\cdot\infty=0$ on $\partial B$) is in $C_c^\infty(\mathbb{R}^N)$.*

**PROOF.** The trick to minimize the computational effort is to decompose $\eta$ under the form $\eta=e_1\circ h$, with $e_1\colon y\in\mathbb{R}\mapsto\exp(-1/(1-y))\chi_{(-\infty,1)}(y)$ and $h\colon x\in\mathbb{R}^N\mapsto|x|^2\in\mathbb{R}$. Note that

Note that $\chi_{(-\infty,1)}(|x|^2)=1$ if, and only if, $|x|^2\leqslant 1$.

$$e_1(x):=\begin{cases}\exp(-1/(1-y)) & \textbf{if } y<1,\\ 0 & \textbf{if } y\geqslant 1.\end{cases}$$

It is then clear that $e_1$ is continuous in $\mathbb{R}$. It is also clear that $\eta$ is continuous on $\mathbb{R}^N$ and $\operatorname{supp}_\Omega\eta=B$.

Next, since $h$ is in $C^\infty(\mathbb{R}^N)$ function (in fact, it is a polynomial), for $\eta$ to be in $C^\infty(\mathbb{R}^N)$ it is sufficient that $e_1\in C^\infty(\mathbb{R})$ because then the assertion follows from the chain rule. In that regard, as $h$ is of class $C^\infty$ in $\mathbb{R}\backslash\{1\}$ we rely on a well-known corollary of de l'Hôpital's rule. Precisely, it is simple to show, by induction, that for every $k\in\mathbb{N}$ there exists a polynomial function $P_k\colon\mathbb{R}\to\mathbb{R}$, such that for every $y<1$

The polynomial $P_k$ will be of order $2k$, but this information is useless for our purposes

$$\partial_y^k e_1(y)=P_k\left(\frac{1}{1-y}\right)e_1(y).$$

On the other hand, it is clear that $\partial_y^k e_1(y)=0$ for any $y>1$ and any $k\in\mathbb{N}$. But then, according to de l'Hôpital's, as $e_1$ is continuous at $y=1$, for $k=1$, we infer that $(\partial_y e_1)_{|y:=(1^-)}=\lim_{y\to1^-}P_1\left(\frac{1}{1-y}\right)e_1(y)=0=(\partial_y e_1)_{|y:=(1^+)}$. Hence, $e_1\in C^1(\mathbb{R})$. Proceeding by induction, one shows that

$$(\partial_y^k e_1)_{|y:=(1^-)}=\lim_{y\to1^-}P_k\left(\frac{1}{1-y}\right)e_1(y)=0=(\partial_y^k e_1)_{|y:=(1^+)}.$$

Therefore, $e_1\in C^\infty(\mathbb{R})$. This concludes the proof.

### 6.3.2. The space $\mathfrak{D}_K^k(\Omega)$ with $k\in\bar{\mathbb{N}}$, $\Omega\subseteq\mathbb{R}^N$ open and $K\subset\Omega$ compact

Let $\Omega\subseteq\mathbb{R}^N$ be an open set, $K\in\mathfrak{K}_\Omega$ and $k\in\bar{\mathbb{N}}$. We denote by $\mathfrak{D}_K^k(\Omega)$ the locally convex space $(C_K^k(\Omega),\tau_K^k)$ with $\tau_K^k:=\tau_\Omega^k|C_K^k(\Omega)$. Recall that $\tau_\Omega^k$ is the topology of $\boldsymbol{\mathcal{E}}^k(\Omega)=(C^k(\Omega),\tau_\Omega^k)$. In other words, the space $\mathcal{D}_K^k(\Omega)$ is the vector subspace $C_K^k(\Omega) \trianglelefteq C^k(\Omega)$ endowed with the topology induced by $\mathcal{E}^k(\Omega) = (C^k(\Omega), \tau_\Omega^k)$.

It follows, from (6.25), that a generalized sequence $(f_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{D}_K^k(\Omega)$ converges to $f \in \mathcal{D}_K^k(\Omega)$ if, and only if, $\mathfrak{p}_{K,m}(f - f_\lambda) := \sup_{|\alpha| \leqslant m} \sup_{x \in K} |\mathsf{D}^\alpha f_\lambda - \mathsf{D}^\alpha f(x)|$ converges to zero for every $m \leqslant k$. This means that for every $|\alpha| \leqslant k$ the generalized sequence $\mathsf{D}^\alpha f_\lambda$ converges uniformly to $\mathsf{D}^\alpha f$ on the compact set $K$.

**6.54. Proposition.** *The space $\mathcal{D}_K^k(\Omega)$ is a Frechét space.*

**Proof.** Note that the notion of first-countable space and Hasusdorff separated space are hereditary, that is, inherited by topological subspaces. Since $\mathcal{E}^k(\Omega)$ is a Frechét space, it remains to show that $\mathcal{D}_K^k(\Omega)$ is complete. For that, it is sufficient to prove that $C_K^k(\Omega)$ is closed in the complete space $\mathcal{E}^k(\Omega)$. To this end, consider a **sequence** $(f_n)_{n \in \mathbb{N}}$ in $C_K^k(\Omega)$ converging, in $\mathcal{E}^k(\Omega)$, towards an element $f \in C^k(\Omega)$. If $x_0 \notin K$ then $f_n(x_0) = 0$ for every $n \in \mathbb{N}$ and therefore $f(x_0) = 0$. Hence, the domain of nullity $U_\Omega(f)$ of $f$ is such that $U_\Omega(f) \supseteq \Omega \backslash K$, that is $\mathrm{supp}_\Omega f \subseteq K$. ■

The domain of nullity is defined in **Definition 6.15**

### 6.3.3. The space $\mathcal{D}^k(\Omega)$ with $k \in \bar{\mathbb{N}}$, $\Omega \subseteq \mathbb{R}^N$ open

Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $k \in \bar{\mathbb{N}}$. In **Definition 6.51** we have seen that $C_c^k(\Omega) \equiv \bigcup_{j \in \mathbb{N}} C_{K_j}^k(\Omega)$, where $K_j := \bar{\Omega}_j$ and $(\Omega_j)_{j \in \mathbb{N}}$ is a sequence of open and relatively compact sets, covering $\Omega$ and such that, for every $j \in \mathbb{N}$, $\bar{\Omega}_j \subseteq \Omega_{j+1}$. Now, recall the *compatibility result* among induced topologies: in general, if $S$ is a topological space, and $A \subseteq B \subseteq S$, the subspace topology that $A$ inherits from the subspace $B$ (endowed with the topology induced by $S$) is the same as the one it inherits from $S$. Therefore, the following statement hold:

*i.* For every $j \in \mathbb{N}$ one has $C_{K_j}^k(\Omega) \trianglelefteq C_{K_{j+1}}^k(\Omega)$ and moreover $C_c^k(\Omega) \equiv \bigcup_{j \in \mathbb{N}} C_{K_j}^k(\Omega)$.

*ii.* The topology of $\mathcal{D}_{K_j}^k(\Omega) = (C_{K_j}^k(\Omega), \tau_{K_j}^k)$ coincides with the topology induced on $C_{K_j}^k(\Omega)$ by $\mathcal{D}_{K_{j+1}}^k(\Omega) = (C_{K_{j+1}}^k(\Omega), \tau_{K_{j+1}}^k)$, because both of them are inherited by $\mathcal{E}^k(\Omega)$.

But this is the general setting of a strict inductive limit of Frechét spaces (cf. **Section 20**).

**6.55. Definition.** The natural topology on $C_c^k(\Omega)$ is, by definition, the topology $\tau_{\mathrm{LF}}^k$ **strict inductive limit** of the sequence of Fréchet spaces $(\mathcal{D}_{K_j}^k(\Omega))_{j \in \mathbb{N}} = (C_{K_j}^k(\Omega), \tau_{K_j}^k)_{j \in \mathbb{N}}$. When endowed with its natural topology, the LF space $(C_c^k(\Omega), \tau_{\mathrm{LF}}^k)$ is denoted by the symbol $\mathcal{D}^k(\Omega)$.

According to **Proposition 5.15**, the topology of $\mathcal{D}_{K_j}^k(\Omega)$ coincides with the topology induced on $C_{K_j}^k(\Omega)$ by $\mathcal{D}^k(\Omega)$. Moreover, since every compact set $K \in \mathfrak{K}_\Omega$ is included in some $K_{j\star}$ of $(K_j := \bar{\Omega}_j)_{j \in \mathbb{N}}$, the following result holds.

**6.56. Proposition.** *For any $K \in \mathfrak{K}_\Omega$ the topology of $\mathcal{D}_K^k(\Omega)$ coincides with the topology induced on $C_K^k(\Omega)$ by $\mathcal{D}^k(\Omega)$.*

**Proof.** For any compact set $K \in \mathfrak{K}_\Omega$ we have that

$$\tau_K^k := \tau_\Omega^k | C_K^k(\Omega).$$

Therefore, if $K_{j\star} \supseteq K$ we have that $C_K^k(\Omega) \subseteq C_{K_{j\star}}^k(\Omega)$ and $\tau_{K_{j\star}}^k := \tau_\Omega^k | C_{K_{j\star}}^k(\Omega)$. Therefore, by the *compatibility result* among induced topologies, $\tau_K^k \equiv \tau_{K_{j\star}}^k | C_K^k(\Omega)$. Eventually, by **Proposition 5.15**, the topology of $\mathfrak{D}_{K_j}^k(\Omega)$ coincides with the topology induced on $C_{K_j}^k(\Omega)$ by $\mathfrak{D}^k(\Omega)$. Hence, $\tau_K^k \equiv \tau_{K_{j\star}}^k | C_K^k(\Omega) \equiv \tau_{K_{j\star}}^k | C_c^k(\Omega)$.

**6.57. Corollary.** *The* LF *topology of* $\mathfrak{D}^k(\Omega)$ *does not depend on the covering sequence* $(\Omega_j)_{j\in\mathbb{N}}$.

**PROOF.** Let $K_j := \bar{\Omega}_j$ and $J_j := \bar{\Gamma}_j$, where $(\Omega_j)_{j\in\mathbb{N}}$ and $(\Gamma_j)_{j\in\mathbb{N}}$ are sequences of open and relatively compact sets, covering $\Omega$ and such that, for every $j \in \mathbb{N}$, $\bar{\Omega}_j \subseteq \Omega_{j+1}$ and $\bar{\Gamma}_j \subseteq \Gamma_{j+1}$. Let $\tau_{\mathrm{LF}}^k$ denote the LF topology induced on $C_c^k(\Omega)$ by $(\mathfrak{D}_{K_j}^k(\Omega))_{j\in\mathbb{N}}$, and denote by $\sigma_{\mathrm{LF}}^k$ the LF topology induced on $C_c^k(\Omega)$ by $(\mathfrak{D}_{J_j}^k(\Omega))_{j\in\mathbb{N}}$. It is sufficient to show that the canonical injection

$$\iota\colon (C_c^k(\Omega), \tau_{\mathrm{LF}}^k) \hookrightarrow (C_c^k(\Omega), \sigma_{\mathrm{LF}}^k)$$

is continuous. Because, after all, we can always switch the roles of $\tau_{\mathrm{LF}}^k$ and $\sigma_{\mathrm{LF}}^k$.

To this end, we observe that as a consequence of **Proposition 5.13** we get that $\iota$ is continuous on $(C_c^k(\Omega), \tau)$ if, and only if, for every compact $K \in \mathfrak{K}_\Omega$ the restriction of $\iota$ to $\mathfrak{D}_K^k(\Omega)$ is continuous (cf. also **Proposition 6.59**). But the map

$$x \in \mathfrak{D}_K^k(\Omega) \mapsto x \in (C_c^k(\Omega), \sigma)$$

is continuous because on $C_K^k(\Omega)$ the topologies induced by $\tau_{\mathrm{LF}}^k$ and $\sigma_{\mathrm{LF}}^k$ coincide with the topology of $\mathfrak{D}_K^k(\Omega)$.

### 6.3.4. Characterization of bounded subsets of $\mathfrak{D}^k(\Omega)$ and convergence in $\mathfrak{D}^k(\Omega)$

As an immediate particularization of Dieudonné-Schwartz theorem (**Theorem 5.19**), more precisely of **Corollary 5.20** and **Corollary 5.21**, the following result holds.

**6.58. Proposition.** *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set,* $k \in \bar{\mathbb{N}}$. *A subset* $B^k(\Omega)$ *of* $\mathfrak{D}^k(\Omega)$ *is* **bounded**, **if**, **and only if**, *there exists a compact subset* $K \in \mathfrak{K}_\Omega$ *such that* $B^k(\Omega) \subseteq \mathfrak{D}_K^k(\Omega)$ *and* $B^k(\Omega)$ *is bounded in* $\mathfrak{D}_K^k(\Omega)$. *Namely:*

$$\big(\operatorname{supp}_\Omega \varphi \subseteq K \quad \forall \varphi \in B^k(\Omega)\big) \quad \textbf{and} \quad \left(\sup_{\varphi \in B^k(\Omega)} \mathfrak{p}_{K,m}(\varphi) < \infty \quad \forall m \leqslant k\right).$$

*A sequence* $(\varphi_n)_{n\in\mathbb{N}}$ *of elements of* $\mathfrak{D}^k(\Omega)$ **converges** *in* $\mathfrak{D}^k(\Omega)$ *towards an element* $\varphi \in \mathfrak{D}^k(\Omega)$, **if**, **and only if** *there exists a compact subset* $K \in \mathfrak{K}_\Omega$ *such that* $\{\varphi, \varphi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}_K^k(\Omega)$ *and* $\varphi_n \to \varphi$ *in* $\mathfrak{D}_K^k(\Omega)$. *Namely:*

- $\operatorname{supp}_\Omega \varphi_n \subseteq K$ *for every* $n \in \mathbb{N}$ *and* $\operatorname{supp}_\Omega \varphi \subseteq K$;
- *for every multi-index* $|\alpha| \leqslant k$ *the sequence* $(\mathsf{D}^\alpha \varphi_n)_{n\in\mathbb{N}}$ *uniformly converges to* $\mathsf{D}^\alpha \varphi$ *in* $K$.

### 6.3.5. Characterization of linear maps $\mathfrak{D}^k(\Omega) \to \mathfrak{Y}$ with $\mathfrak{Y}$ locally convex space

As usual, let $\Omega \subseteq \mathbb{R}^N$ be an open set, $k \in \bar{\mathbb{N}}$.

**6.59. Proposition.** *Let $T$ be a linear map of $\mathcal{D}^k(\Omega)$ into a* **locally convex space** *(in particular, a linear form). The following three assertions are equivalent:*

*i. $T$ is continuous.*

*ii. $T$ is sequentially continuous.*

*iii. For every compact $K$ in $\Omega$, the restriction of $T$ to $\mathcal{D}_K^k(\Omega)$ is (sequentially) continuous.*

**PROOF.** The equivalence of *i.* and *ii.* is a specialization of **Proposition 5.18**, which holds for LF spaces. The equivalence of *ii.* and *iii.* is a particularization of **Proposition 5.13** as soon as we note that every compact subset $K\in\mathfrak{K}_\Omega$ is included in some $K_{j\star}$ of $(K_j:=\bar{\Omega}_j)_{j\in\mathbb{N}}$.

### 6.3.6. Order relations for the topology of $\mathcal{D}^k(\Omega)$

**6.60. Proposition.** *For every $k\in\bar{\mathbb{N}}$, the topology of $\mathcal{D}^k(\Omega)$ is finer (stronger) than the topology induced on $C_c^k(\Omega)$ by $\mathcal{E}^k(\Omega)$. For every $k,h\in\bar{\mathbb{N}}$, if $h\geqslant k$ then the topology of $\mathcal{D}^h(\Omega)$ is finer (stronger) than the subspace topology induced on $C_c^h(\Omega)$ by $\mathcal{D}^k(\Omega)$. Also, for every $p\in[1,\infty]$ the topology of $\mathcal{D}^k(\Omega)$ is finer (stronger) than the one induced by the Lebesgue spaces $L^p(\Omega)$.*

**PROOF.** Let us show that the canonical injection of $\mathcal{D}^k(\Omega)$ into $\mathcal{E}^k(\Omega)$ is continuous. According to **Proposition 6.59** it is sufficient to show that the injection of $\mathcal{D}_K^k(\Omega)$ into $\mathcal{E}^k(\Omega)$ is continuous for every $K\in\mathfrak{K}_\Omega$. But this is trivial because the topology of $\mathcal{D}_K^k(\Omega)$ is the one induced by $\mathcal{E}^k(\Omega)$ on $C_K^k(\Omega)$.

Similarly, the canonical injection of $\mathcal{D}^h(\Omega)$ into $\mathcal{D}^k(\Omega)$ is continuous. As before, it is sufficient to show that the injection of $\mathcal{D}_K^h(\Omega)$ into $\mathcal{D}^k(\Omega)$ is sequentially continuous for every $K\in\mathfrak{K}_\Omega$. But this is an easy consequence of **Theorem 6.58**.

In the same way, one realizes that the canonical injection of $\mathcal{D}^k(\Omega)$ into $L^p(\Omega)$ is continuous for every $p\in[1,\infty]$.

### 6.3.7. Dense subspaces of $\mathcal{D}^k(\Omega)$

**6.61. Theorem.** *Let $k\in\bar{\mathbb{N}}$. The space $C_c^\infty(\Omega)$, considered as a vector subspace of $\mathcal{D}^k(\Omega)$, is dense in $\mathcal{D}^k(\Omega)=(C_c^k(\Omega),\tau_{\mathrm{LF}}^k)$. It follows that $C_c^h(\Omega)$ is dense in $\mathcal{D}^k(\Omega)$ for any $h,k\in\bar{\mathbb{N}}$.*

### 6.3.8. Dense subspaces of $\mathcal{E}^k(\Omega)$

**6.62. Theorem.** *Let $k\in\bar{\mathbb{N}}$. The space $C_c^\infty(\Omega)$, considered as a vector subspace of $\mathcal{E}^k(\Omega)$, is dense in $\mathcal{E}^k(\Omega)=(C_c^k(\Omega),\tau_\Omega^k)$. It follows that $C_c^h(\Omega)$ is dense in $\mathcal{E}^k(\Omega)$ for any $h,k\in\bar{\mathbb{N}}$.*

○

# 7

# Duality

## 7.1 | Dual pairs

### 7.1.1. Duality pairing: non-degeneracy and orthogonality.

A **dual pair** is a triple $(Y, X, \langle\cdot,\cdot\rangle)$ consisting of two vector spaces $Y$ and $X$, over the same field $\mathbb{K}$, and a **bilinear map** $(y,x) \in Y \times X \mapsto \langle y, x\rangle \in \mathbb{K}$ such that the following **non-degeneracy** conditions are satisfied:

$$\forall y \in Y \setminus \{0\} \;\; \langle y, \cdot\rangle \not\equiv \langle 0, \cdot\rangle \quad \textbf{and} \quad \forall x \in X \setminus \{0\} \;\; \langle \cdot, x\rangle \not\equiv \langle \cdot, 0\rangle. \tag{7.1}$$

Note that, by bilinearity, the notation $\langle 0, \cdot\rangle$ (resp. $\langle \cdot, 0\rangle$) is just a shortcut to denote the null functional on $X$ (resp. on $Y$). Equivalently, the bilinear form $\langle\cdot,\cdot\rangle$ satisfies the nondegeneracy condition if, and only if,

If the vector spaces are finite dimensional the two conditions in (7.1) mean that the bilinear form is **non-degenerate**.

$$\forall y \in Y \quad [\langle y, \cdot\rangle = \langle 0, \cdot\rangle \implies y = 0], \tag{7.2}$$
$$\forall x \in X \quad [\langle \cdot, x\rangle = \langle \cdot, 0\rangle \implies x = 0]. \tag{7.3}$$

We call $\langle\cdot,\cdot\rangle$ the **duality pairing** (on $Y \times X$), and we say that the bilinear form $\langle\cdot,\cdot\rangle$ places the vector spaces $X$ and $Y$ in **duality**.

**7.1. Remark.** Note that, although no symmetry assumption on $\langle\cdot,\cdot\rangle$ is imposed by the definition, the notion of duality pair has an intrinsic symmetric character. By this we mean that if $\langle\cdot,\cdot\rangle$ puts $X$ and $Y$ in duality, then the bilinear form on $X \times Y$ defined by $\langle\!\langle x, y\rangle\!\rangle := \langle y, x\rangle$, puts $Y$ and $X$ in duality. Therefore one can say that $\langle\cdot,\cdot\rangle$ places $X$ and $Y$ in duality or that it places $Y$ and $X$ in duality.

A further characterization of the nondegeneracy conditions is stated in the next result.

**7.2. Proposition.** *Let $Y$ and $X$ be two vector spaces and $(Y, X, \langle\cdot,\cdot\rangle)$ a dual pair. The following assertions hold:*

*i. If $\langle y_1, x\rangle = \langle y_2, x\rangle$ for every $x \in X$, then necessarily $y_1 = y_2$.*

*ii. If $\langle y, x_1\rangle = \langle y, x_2\rangle$ for every $y \in Y$, then necessarily $x_1 = x_2$.*

*The previous two conditions are, taken together, equivalent to the* **non-degeneracy** *conditions, although, taken individually, they are more often referred to as* **separating** *conditions.*

**7.3. Remark.** If we denote by $Y^*$ the algebraic dual of $Y$, then condition $i.$ says that the family of linear forms $x \in X \mapsto \langle \cdot, x \rangle \in Y^*$ separates the points of $Y$. Similarly, if we denote by $X^*$ the algebraic dual of $X$, then condition $ii.$ says that the family of linear forms $y \in Y \mapsto \langle y, \cdot \rangle \in X^*$ separates the points of $X$. Recall that a family of linear functional in $Y^*$ separated the points if twhenever $y_1 \neq y_2$ there exists an element $y^* \in Y^*$ that recognize when $y_1, y_2$ are different, i.e., such that $y^*(y_1) \neq y^*(y_2)$. The terminology is borrowed from the one adopted for seminorms because, clearly, $(|\langle \cdot, x \rangle|)_{x \in X}$ (resp. $(|\langle y, \cdot \rangle|)_{y \in Y}$) is a separating family of seminorms on $Y$ (resp. on $X$).

**7.4. Remark.** For every $y \in Y$ the map $\langle y, \cdot \rangle : x \in X \mapsto \langle y, x \rangle \in \mathbb{K}$ defines an element of the algebraic dual $X^*$. Therefore, if we identify the bilinear form $\langle \cdot, \cdot \rangle$ with the linear map

$$L: y \in Y \mapsto \langle y, \cdot \rangle \in X^*$$

then the nondegeneracy condition (7.2) is equivalent to the injectivity of $L$. Therefore, a duality pair permits to identify $Y$ to a subspace of $X^*$. Similarly, the nondegeneracy condition (7.3) guarantees tha the linear map

$$R: x \in X \mapsto \langle \cdot, x \rangle \in Y^*$$

is injective. Therefore, a duality pair allows indentifying $X$ to a subspace of $Y^*$. In general, however, the maps $L$ and $R$ are not algebraic isomorphisms due to the lack of surjectiveness (see Remark ...)

When $Y = X^*$, i.e., when $Y$ is the algebraic dual of $X$, the bilinear form

$$(x^*, x) \in X^* \times X \mapsto \langle x^*, x \rangle = x^*(x) \in \mathbb{K} \tag{7.4}$$

is referred to as the **canonical duality pairing** (or **the natural pairing**) on $X$. The map

$$L: \; x^* \in X^* \mapsto \langle x^*, \cdot \rangle \in X^* \tag{7.5}$$

is nothing but the identity map and, therefore, surjective. On the other hand, the map

$$R: \; x \in X \mapsto \langle x^*, \cdot \rangle \in X^{**} \tag{7.6}$$

is the **canonical map** (or the **natural map**) from $X$ into $X^{**}$ which maps each point $x \in X$ to the **evaluation map at** $x$ defined by $\mathrm{ev}_x : x^* \in X^* \mapsto \langle x^*, x \rangle = x^*(x) \in \mathbb{K}$. Clearly, $\mathrm{ev}_x \in X^{**}$. We recall that $X^{**}$ is the double (algebraic) dual space (also called algebraic bidual) of $X$. In **Proposition 7.7** below we show that the canonical duality pairing sets $X$ and $X^*$ in duality. In particular, the natural map $R$ in (7.6) defines a monomorphism of $X$ into $X^{**}$.

In general, however, the natural map $R$ in (7.6) is not an algebraic isomorphism due to the lack of surjectiveness. In fact, it is possible to show that

$$\dim(X) \leqslant \dim(X^*) \leqslant \dim(X^{**}) \leqslant \dim(X^{***}) \leqslant \ldots \tag{7.7}$$

with equality if, and only if, $X$ is finite-dimensional. In other words, in infinite-dimensional spaces, the duality operator strictly increases the cardinality of the bases. Therefore, there are no chances to find an isomorphism between $X$ and $X^{**}$. By contrast, when $X$ is finite-dimensional, we have $\dim(X) = \dim(X^{**})$ and, therefore, $X$ and $X^{**}$ are isomorphic. However, the existence of at least an isomoprhism in the finite-dimensional setting does not mean that the injective map $R$ in (7.6) has to be among them. But this is indeed the case (the easy proof is omitted), i.e., if $X$ is finite-dimensional then the natural map (7.6) is also surjective.

The dual of an infinite-dimensional space has greater dimensionality (this being a greater infinite cardinality) than the original space has, and thus these cannot have a basis with the same indexing set. However, a dual set of vectors exists, which defines a subspace of the dual isomorphic to the original space.

Overall, if we agree to say that a vector space $X$ is **algebraically reflexive** when the *natural map* is an isomorphism, then we can condensate the previous considerations by saying that a vector space $X$ is algebraically reflexive if, and only if, it is finite-dimensional.

**7.5. Definition.** We say that two **elements** $y \in Y$ and $x \in X$ **are orthogonal** when

$$\langle y, x \rangle = 0. \tag{7.8}$$

We say that two **sets**, $M \subseteq X$ and $N \subseteq Y$, **are orthogonal** if any elements of $M$ is orthogonal to every element of $N$. In other words, $M$ is orthogonal to $N$ if, and only if, $\langle y, x \rangle = 0$ for every $(y, x) \in N \times M$. In symbols:

$$N \perp M \quad \Longleftrightarrow \quad \langle y, x \rangle = 0 \ \ \forall (y, x) \in N \times M. \tag{7.9}$$

If $A \subseteq X$ the set $A^{\perp} := \{y \in Y :: \langle y, x \rangle = 0 \text{ for any } x \in A\}$ is called **the orthogonal of $A$ in $Y$** and, as it is easy to show, it is a vector subspace of $Y$:

$$A^{\perp} \trianglelefteq Y \quad \forall A \in \wp(X). \tag{7.10}$$

Similarly, if $B \subseteq Y$, the set $B^{\perp} := \{x \in X :: \langle y, x \rangle = 0 \text{ for any } y \in B\}$ is called **the orthogonal of $B$ in $X$**, and is a vector subspace of $X$:

$$B^{\perp} \trianglelefteq X \quad \forall B \in \wp(X). \tag{7.11}$$

When $N$ is a singleton, we simplify the notation $\{y\} \perp M$ to $y \perp M$ (and in a similar way must be interpreted the notations $N \perp x$ and $y \perp x$).

**7.6. Remark.** Note that, by the nondegeneracy conditions, we have that for any $y \in Y$ there holds the following equivalence: $y \perp X$ if, and only if, $y = 0$. Similarly, for any $y \in Y$ there holds the following equivalence: $Y \perp x$ if, and only if, $x = 0$.

### 7.1.2. Canonical dual pairs: duality via the algebraic and continuous dual spaces

Recall that, given a vector space $X$, the space $X^*$ denotes its **algebraic dual**, that is, the set of all **linear forms** on $X$ (with values in $\mathbb{K}$). If $\mathfrak{X}$ is a topological vector space, we denote by $\mathfrak{X}'$ the **continuous dual** of $\mathfrak{X}$, that is, the set of all **linear** and **continuous functionals** on $\mathfrak{X}$ (with values in $\mathbb{K}$).

**Note** the different use of the words «functional» and «form». We say a linear form when $\mathbb{K}$ is seen as a purely algebraic vector space. We talk of linear functionals, when $\mathbb{K}$ is endowed with its topological vector space structure.

The main purpose of this section is to prove the following result.

**7.7. Proposition.** *A vector space $X$ together with its* **algebraic** *dual $X^*$ and the bilinear* **pointwise evaluation map** *defined by*

$$\langle f, x \rangle := f(x) \quad \textbf{for any } (x \in X, f \in X^*),$$

*forms a dual pair.*

*Moreover, if $\mathfrak{X}$ is a* **Hausdorff separated** *and* **locally convex** *(topological vector)* **space***, then $\mathfrak{X}$ together with its continuous dual $\mathfrak{X}'$ and the bilinear pointwise evaluation map forms a dual pair. Here*

$$\langle f, x \rangle := f(x) \quad \textbf{for any } (x \in \mathfrak{X}, f \in \mathfrak{X}').$$

Note that, the first condition in (7.1) is trivially satisfied when $Y = X^*$ (resp. when $Y = \mathfrak{X}'$) because it is nothing but the definition of non-zero linear form on $X$ (resp. functional on $\mathfrak{X}'$). On the other hand, to show that the second condition in (7.1) is satisfied requires a proper argument. That is why, before proving Proposition 7.7, we need to recall some basic facts from linear algebra.

**7.8. Proposition.** *Let $M$ be a subspace of a vector space $X$. A linear form $f_0\colon M \to \mathbb{K}$ can always be extended to a linear form $f\colon X \to \mathbb{K}$, so that $f_{|M} \equiv f_0$. For example, if $m\colon X \to M$ is the algebraic projection of $X$ along $M$, then we can set*

$$f(x) = f_0(m(x)).$$

*This is the so-called canonical extension of $f_0$ to $X$.*

We now have the tools to prove that $(X^*, X, \langle\cdot,\cdot\rangle)$ with $\langle\cdot,\cdot\rangle$ the (bilinear) **pointwise evaluation map**, forms a dual pair.

**Proof.** [of **Proposition 7.7**, the **algebraic dual** setting] We have to show that the second condition in (7.1) holds. From **Proposition 7.8**, the following claim holds: *if $0 \neq x_0 \in X$ is a nonzero vector, then there exists a linear form $f\colon X \to \mathbb{K}$ such that $f(x_0) = 1$.*

Indeed, it is sufficient to consider the one-dimensional subspace $M = \{\lambda x_0 :: \lambda \in \mathbb{K}\}$ and then the *canonical extension* to the full space of the linear form $f_0\colon M \to \mathbb{K}$ defined as $f_0(\lambda x_0) = \lambda$, that is, as the coordinate chart of $M$ induced by the basis $(x_0)$.

Therefore, the set of all linear forms on a vector space $X$ separates the points of $X$, meaning that *if $x_0 \in X$ and $f(x_0) = 0$ for any $f \in X^*$ then necessarily $x_0 = 0$*. It is common to refer to this property by saying that the **algebraic** dual $X^*$ of a vector space $X$ separates the points in $X$. ■■■■

The proof of **Proposition 7.7** in the locally convex setting requires more tools. We start by reviewing some basic facts concerning the continuous dual, mainly the Hahn-Banach theorem. This is the object of the next section where also the proof of **Proposition 7.7** will be completed.

### 7.1.3. The Hahn-Banach theorem in purely algebraic vector spaces

Let us recall the classical algebraic form of the Hahn-Banach theorem.

**7.9. Theorem. Assumptions:** *Let $X$ be a vector space on $\mathbb{K}$ ($\mathbb{K}$ equals $\mathbb{R}$ or $\mathbb{C}$), $M \trianglelefteq X$ a vector* **sub***space of $X$, and $\mathfrak{p}$ a seminorm on $X$.*

**Claim:** *Every (algebraic) linear form $\varphi$* **on** *$M$ such that $|\varphi| \leqslant \mathfrak{p}$* **in** *$M$ can be extended to a linear form $\tilde{\varphi}$ on $X$ while preserving the constraint:*

$$|\tilde{\varphi}| \leqslant \mathfrak{p} \quad \textbf{in } X.$$

**Claim in symbols (* denotes the algebraic dual):** *for every $\varphi \in M^*$, satisfying the constraint $|\varphi| \leqslant \mathfrak{p}_{|M}$, there exists $\tilde{\varphi} \in X^*$ such that $\tilde{\varphi}_{|M} \equiv \varphi$ and $|\tilde{\varphi}| \leqslant \mathfrak{p}$* **in** *$X$.*

**7.10. Remark.** Note that, in order to use the general algebraic form of the Hahn-Banach theorem, one needs, apriori, a seminorm $\mathfrak{p}$ defined **all over** the space $X$ (although the linear form we are going to extend is defined just on a subspace). When $\mathfrak{X}$ is a locally convex space, this bothering assumption is (in some sense) given for free (cf. **Theorem 7.11**).

### 7.1.4. The Hahn-Banach theorem in a locally convex space

It is possible to prove that the topological dual $\mathfrak{X}'$ of a topological vector space $\mathfrak{X} \neq \{0\}$ may consist of the zero functional only (cf. Example K in HANS JARCHOW, *Locally Convex Spaces*, Teubner

Stuttgart **1981**, p. 123). On the other hand, if $\mathfrak{X}$ is a Hausdorff separated locally convex space, then $\mathfrak{X}'$ always contains sufficiently many elements for a meaningful duality theory.

**7.11. Theorem. Assumptions:** *Let $\mathfrak{X}$ be a* **locally convex** *(topological vector) space on $\mathbb{K}$, $\mathfrak{M} \trianglelefteq \mathfrak{X}$ a (topological vector)* **sub***space of $\mathfrak{X}$, and $f$ a* **continuous** *linear functional on $\mathfrak{M}$.*

**Claim:** *There exists a* **continuous** *linear functional $\tilde{f}$ defined on $\mathfrak{X}$ which extends $f$.*

**Claim in symbols:** *for every $f \in \mathfrak{M}'$, there exists $\tilde{f} \in \mathfrak{X}'$ such that $\tilde{f}_{|\mathfrak{M}} \equiv f$.*

**Corollary:** *If $\mathfrak{X}$ is also Hausdorff separated, then for every $\mathfrak{X} \ni x_0 \neq 0$ there exists a continuous linear functional $f$ on $\mathfrak{X}$ such that $f(x_0) \neq 0$.*

Therefore, also such that $f(x_0) = 1$. If $f$ is such that $f(x_0) \neq 0$, just redefine $f(x) := \frac{f(x)}{f(x_0)}$.

**7.12. Remark.** The corollary stated in **Proposition 7.11** is almost trivial in a finite-dimensional setting as any linear functional defined in a finite-dimensional and Hausdorff separated topological vector space is continuous. Precisely, relying on **Theorem 4.51**, it is simple to prove that: *if $\mathfrak{X}$ is a Hausdorff separated, finite-dimensional, topological vector space and $\mathfrak{Y}$ a topological vector space (not necessarily Hausdorff separated nor finite-dimensional), then any linear map from $\mathfrak{X}$ into $\mathfrak{Y}$ is continuous.*

### 7.1.5. Dual pairs in locally convex spaces

When $\mathfrak{X}$ is a locally convex space, it is sufficient to consider just its **continuous** dual space $\mathfrak{X}'$ (included in its algebraic dual $\mathfrak{X}^*$) in order to separate the points of $\mathfrak{X}$. This is a consequence of the Hahn-Banach theorem for Hausdorff separated locally convex spaces.

**PROOF.** [of **Proposition 7.7**, the **continuous dual** setting] It is nothing but an equivalent restatement of the corollary in **Proposition 7.11**: *if $x_0 \in \mathfrak{X}$ and $f(x_0) = 0$ for any $f \in \mathfrak{X}'$ then necessarily $x_0 = 0$.* Thus, $\mathfrak{X}'$ separates the points of $\mathfrak{X}$.

## 7.2 The polar set

Let $\mathfrak{X}$ be a topological vector space on $\mathbb{K}$, $\mathfrak{X}'$ its continuous dual space, that is, the set of all linear and continuous functionals on $\mathfrak{X}$. Let $A$ be **any** non-empty subset of $\mathfrak{X}$.

**7.13. Definition.** We define the **polar set** of $A$ as the subset of $\mathfrak{X}'$ given by

$$A^\circ := \left\{ x' \in \mathfrak{X}' :: \sup_{x \in A} |\langle x', x \rangle| \leqslant 1 \right\}. \tag{7.15}$$

Note that the polar set $A^\circ$ of a (*non-empty*) subset $A$ always contains the **null functional**.

Recall that the empty set is always bounded (cf. remark 3.40). Also, recall that the supremum in $\mathbb{R}$ of $\emptyset$ is $+\infty$.

**7.14. Remark.** Note that, in general, the functional $x' \in \mathfrak{X}' \mapsto \sup_{x \in A} |\langle x', x \rangle|$ is not a seminorm because the value $\sup_{x \in A} |\langle x', x \rangle|$ can be infinite. Also, as already observed at the beginning of **Section 7.1.4**, if $\mathfrak{X}$ is just a topological vector space (*not locally convex*), it can happen that $\mathfrak{X}'$ reduces to the null functional. In this case, $A^\circ = \{0 \in \mathfrak{X}'\}$ for any $A \subseteq \mathfrak{X}$.

Also, note that if $x' \in A^\circ$ then also $-x' \in A^\circ$. More generally, as we are going to show, the polar set of $A$ is an absorbing, balanced, and convex set.

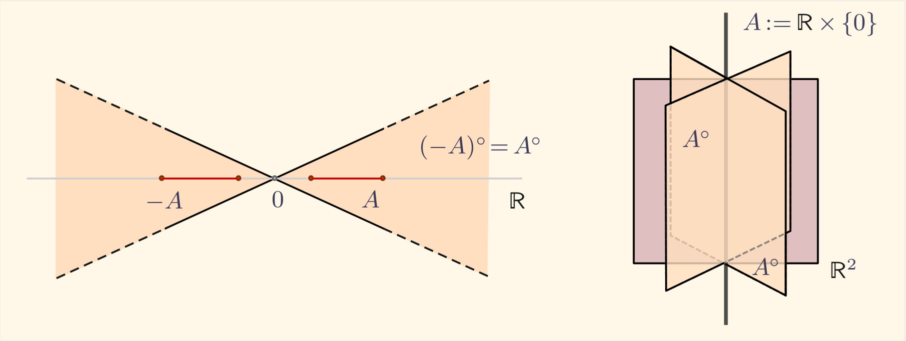



**Figure 7.1.** **(Left)** If $x' \in (\rho A)^\circ$ then $|\rho| x' \in A^\circ$. In particular, $(-A)^\circ = A^\circ$. **(Right)** If $A$ is a vector **sub**space of $\mathfrak{X}$ then $A^\circ$ coincides with the **orthogonal of** $A$.

Immediate consequences of the definition are collected in the following result.

**7.15. Proposition.** *The following properties depend only on the algebraic vector space structure of* $\mathfrak{X}$.

**a)** *Let $A, B$ be subsets of $\mathfrak{X}$ and $\rho \neq 0$, $\rho \in \mathbb{K}$. The following properties hold:*

$$\textit{i.}\ \ A \subseteq B \Rightarrow B^\circ \subseteq A^\circ; \quad \textit{ii.}\ \ (A \cup B)^\circ = A^\circ \cap B^\circ; \quad \textit{iii.}\ \ (\rho A)^\circ = \frac{1}{|\rho|} A^\circ$$

**b)** *If $A$ is a vector* **sub***space of $\mathfrak{X}$ then $A^\circ$ coincides with the* **orthogonal of** *$A$, that is, with the vector subspace of $\mathfrak{X}'$* (cf. **Figure 7.1**)

$$A^\perp := \{x' \in \mathfrak{X}' :: \langle x', x\rangle = 0 \ \textbf{for every} \ \ x \in A\}.$$

**c)** *The polar set $A^\circ$ is always a convex and balanced subset of $\mathfrak{X}'$.*

**Note:** *Relation* **a).iii.** *means that for any $x' \in \mathfrak{X}'$, if $x' \in (\rho A)^\circ$ then $|\rho| x' \in A^\circ$. In particular,* $(-A)^\circ = A^\circ$ (cf. **Figure 7.1**).

The next result relies on the topological structure of $\mathfrak{X}$. This is a good moment to remind some notation and definitions already presented in the previous chapters.

**Remainder of Definition 1.50 on the gauge of a set:** Let $X$ be a vector space and $A \subseteq X$. The map $\mathfrak{p}_A : X \to [0, +\infty]$ defined, for any $x \in X$, by $\mathfrak{p}_A(x) := \inf\{\alpha \in \mathbb{R}_+^* :: x \in \alpha A\}$ with $\mathbb{R}_+^* = ]0, +\infty[$, is called the **gauge of** $A$ (or **the Minkowski functional induced by** $A$). Here, we assume the usual convention $\inf \emptyset = +\infty$.

**7.16. Proposition.** *Let $\mathfrak{X}$ a topological vector space. If $A$ is* **bounded** *(and non-empty) then $A^\circ$ is absorbing. Moreover, the gauge $\mathfrak{p}_{A^\circ}$ of $A^\circ$ is a seminorm and has the expression*

$$\mathfrak{p}_{A^\circ}(x') \equiv \mathfrak{p}_A(x') := \sup_{x \in A} |\langle x', x\rangle|. \tag{7.16}$$

*In other words, if $A$ is bounded (and non-empty), then $\mathfrak{p}_{A^\circ}(x') := \inf\{\alpha \in \mathbb{R}_+^* :: x' \in \alpha A^\circ\}$ is a seminorm which coincides with the map*

$$\mathfrak{p}_A : x' \in \mathfrak{X}' \mapsto \sup_{x \in A} |\langle x', x\rangle|.$$

Note the subscript: $\mathfrak{p}_{A^\circ}$ is the gauge while $\mathfrak{p}_A$ is the seminorm $x' \mapsto \sup_{x\in A} |\langle x', x\rangle|$ that, as we are going to show, coincides with the gauge.

**Note 7.17.** Note that with a small abuse of notation, we denoted by $\mathfrak{p}_A$ the seminorm $x' \in$

$\mathfrak{X}' \mapsto \sup_{x\in A} |\langle x', x\rangle|$ and not the gauge associated with $A$.

**Proof.** First, we prove that $A^\circ$ is absorbing. Let $A$ be a bounded subset of $\mathfrak{X}$. Consider a generic element $x' \in \mathfrak{X}'$. Since the image of a bounded set via $x'$ is still a bounded subset (cf. **Proposition 3.48**), the subset of real numbers $\{|\langle x', x\rangle| :: x \in A\}$ is a bounded set. Thus, there exists $\alpha > 0$, depending on $x'$, such that $\sup_{x\in A} |\langle x', x\rangle| \leqslant \alpha$. Hence, $x' \in \alpha A^\circ$. As $A^\circ$ is balanced, we conclude that $A$ is absorbing.

Next, we are going to make use of **Proposition 1.51** and of **Corollary 1.53**. Indeed, since $A^\circ$ is balanced, convex and absorbing, the gauge $\mathfrak{p}_{A^\circ}$ is a seminorm on $\mathfrak{X}'$, and although in general $B_\circ(\mathfrak{p}_{A^\circ}) \subseteq A^\circ \subseteq B_\bullet(\mathfrak{p}_{A^\circ})$, here we have $A^\circ = B_\bullet(\mathfrak{p}_{A^\circ})$ because

$$\begin{aligned} B_\bullet(\mathfrak{p}_{A^\circ}) &= \{x' \in \mathfrak{X}' :: \mathfrak{p}_{A^\circ}(x') \leqslant 1\} \\ &= \{x' \in \mathfrak{X}' :: \inf\{\alpha \in \mathbb{R}_+^* :: x' \in \alpha A^\circ\} \leqslant 1\} \\ &= \{x' \in \mathfrak{X}' :: \inf\{\alpha \in \mathbb{R}_+^* :: \sup_{x\in A} |\langle x', x\rangle| \leqslant \alpha\} \leqslant 1\} \\ &= \{x' \in \mathfrak{X}' :: \sup_{x\in A} |\langle x', x\rangle| \leqslant 1\} \\ &= A^\circ. \end{aligned}$$

Hence, if we prove that

$$\mathfrak{p}_A(x') := \sup_{x\in A} |\langle x', x\rangle|,$$

defines a seminorm on $\mathfrak{X}'$, then obviously $B_\bullet(\mathfrak{p}_{A^\circ}) = B_\bullet(\mathfrak{p}_A)$ and therefore, according to **Corollary 1.53**, we get $\mathfrak{p}_A \equiv \mathfrak{p}_{A^\circ}$. Let us prove that this is indeed the case. That $\mathfrak{p}_A(x') < \infty$ it follows from the fact that $A$ is bounded (and non-empty). The circular homogenéity follows from the relation $|\langle \lambda x', x\rangle| = |\lambda|\,|\langle x', x\rangle|$ and the well-known fact that $\sup(\lambda f) = \lambda \sup f$ for any real-valued function $f$ and any $\lambda > 0$. Finally, the additivity results from the triangular inequality

$$|\langle x_1' + x_2', x\rangle| \leqslant |\langle x_1', x\rangle| + |\langle x_2', x\rangle|$$

and the fact that $\sup(f+g) \leqslant \sup f + \sup g$. This completes the proof.

**7.18. Remark.** In the proof, we also showed that, although in general $B_\circ(\mathfrak{p}_A) \subseteq A^\circ \subseteq B_\bullet(\mathfrak{p}_A)$, here we have $B_\bullet(\mathfrak{p}_A) = A^\circ$.

## 7.3 Topologies on the dual space

Let $\mathfrak{X}$ be a topological vector space, and $\mathfrak{S}$ a **filtered by inclusion** family of bounded (non-empty) subsets of $\mathfrak{X}$, i.e., a family of bounded subsets of $\mathfrak{X}$ directed by the relation $\subseteq$. In other terms, $\mathfrak{S}$ is a family of bounded subsets of $\mathfrak{X}$ such that for any pair $(S_1, S_2)$ of bounded subsets in $\mathfrak{S}$ there exists a bounded subset $S \in \mathfrak{S}$ such that $S_1 \cup S_2 \subseteq S$.

**Example 7.19.** As the union of a finite number of bounded sets is bounded (cf. **Proposition 3.44**), the family $\mathfrak{S}$ of **all** bounded (and non empty) subsets of $\mathfrak{X}$ is filtered by inclusion. Indeed for any pair $(S_1, S_2)$ of bounded subsets of $\mathfrak{X}$, there exists $S \in \mathfrak{S}$, such that $S_1 \cup S_2 = S$.

**Example 7.20.** As any finite (cardinality) subset of $\mathfrak{X}$ is bounded (cf. **Proposition 3.44**), the family $\mathfrak{S}$ of all finite subsets of $\mathfrak{X}$ is a filtered-by-inclusion family of bounded subsets of $\mathfrak{X}$.

According to **Proposition 7.16**, if $S$ is a **bounded** (and non-empty) subset (in particular an element of $\mathfrak{S}$) then $S^\circ$ is absorbing, and the gauge $\mathfrak{p}_{S^\circ}$ of $S^\circ$ is a seminorm that has the expression

$\mathfrak{p}_{S^\circ} \equiv \mathfrak{p}_S(x') := \sup_{x\in S} |\langle x', x\rangle|$. Moreover, $S^\circ$ is the closed unit semiball $B_\bullet(\mathfrak{p}_S)$ of $\mathfrak{p}_S$. This justifies the following definition.

**7.21. Definition.** We call $\mathfrak{S}$-topology on $\mathfrak{X}'$ the **locally convex** topology defined by taking the family $(\mathfrak{p}_S)_{S\in\mathfrak{S}}$ as a basis of (continuous) seminorms. In other words, the $\mathfrak{S}$-topology on $\mathfrak{X}'$ is the topology defined by considering

$$\mathcal{B}_{\mathfrak{S}} := \{\rho S^\circ :: S \in \mathfrak{S}, \rho > 0\} = \{\rho B_\bullet(\mathfrak{p}_S) :: S \in \mathfrak{S}, \rho > 0\}$$

as a filter base of neighborhoods **of the origin** of $\mathfrak{X}'$ (recall that the polar of a non empty set always contains the **null functional**).

Note the fundamental role played by *polar set*. They permit to endow the continuous dual of any topological vector space – which is naturally endowed with just the algebraic vector space structure of (pointwise) addition of linear functionals and multiplication of a linear functional by a scalar – with a *locally convex* topology.

**7.22. Remark.** Observe that (cf. Proposition 7.15) for any $\rho > 0$

$$\rho S^\circ = (\rho^{-1}S)^\circ \quad = \quad \left\{x' \in \mathfrak{X}' :: \sup_{x\in S} |\langle x', \rho^{-1}x\rangle| \leqslant 1\right\} = \left\{x' \in \mathfrak{X}' :: \sup_{x\in S} |\langle x', x\rangle| \leqslant \rho\right\}.$$

Hence

$$\mathcal{B}_{\mathfrak{S}} = \{x' \in \mathfrak{X}' :: \mathfrak{p}_S(x') \leqslant \rho\}_{(\rho,S)\in\mathbb{R}^+\times\mathfrak{S}}.$$

Also, note that $\mathfrak{S}^\circ \equiv \{S^\circ :: S \in \mathfrak{S}\}$ is a filter base because $S \supseteq S_1 \cup S_2$ implies (cf. Proposition 7.15) that $S^\circ \subseteq (S_1 \cup S_2)^\circ = S_1^\circ \cap S_2^\circ$. Moreover, $(\mathfrak{p}_S)_{S\in\mathfrak{S}}$ is a filtering family of seminorms because $S \supseteq S_1 \cup S_2$ implies $\mathfrak{p}_S \succcurlyeq \mathfrak{p}_{S_1} \vee \mathfrak{p}_{S_2}$.

The straightforward consequences of the definition are collected in the following result.

**7.23. Proposition.** *Let $\mathfrak{X}$ be a topological vector space. The following assertions hold:*

*a)* *The $\mathfrak{S}$-topology on $\mathfrak{X}'$ is (Hausdorff) separated* **when** *$\mathfrak{S}$ covers the whole space $\mathfrak{X}$.*

*b)* *A generalized sequence $(x'_\lambda)$ converges to $x' \in \mathfrak{X}'$ for the $\mathfrak{S}$-topology* **if**, **and only if**, *the generalized sequence $(x'_\lambda)$ converges to $x'$* **uniformly** *on every $S \in \mathfrak{S}$.*

PROOF. *a)* Since the $\mathfrak{S}$-topology on $\mathfrak{X}'$ is a locally convex topology defined by the basis of continuous seminorms $(\mathfrak{p}_S)_{S\in\mathfrak{S}}$, it is sufficient to prove that for any $x' \neq 0$ there exists $S \in \mathfrak{S}$ such that $\mathfrak{p}_S(x') \neq 0$.

To this end, let $x' \not\equiv 0$. This means, by the very definition, that $\langle x', x_0\rangle \neq 0$ for some $x_0 \in \mathfrak{X}$. Now, by hypothesis, the set $\mathfrak{S}$ covers $\mathfrak{X}$, and therefore there exists $S_0 \in \mathfrak{S}$ such that $x_0 \in S_0$. But then $0 \neq |\langle x', x_0\rangle| \leqslant \mathfrak{p}_{S_0}(x')$.

*b)* The property immediately follows from the relation

$$\mathfrak{p}_S(x'_\lambda) = \sup_{x\in S} |\langle x'_\lambda, x\rangle|$$

and from the characterization of convergence of generalized sequences in locally convex spaces defined by a family of seminorms (cf. Proposition 4.34).

### 7.3.1. Natural topologies on the dual space

**7.24. Definition.** We call **weak dual topology** (or **weak-∗ topology**) on $\mathfrak{X}'$, the (locally convex) $\mathfrak{S}$-topology on $\mathfrak{X}'$ defined by taking as filtered by inclusion family of (non-empty) bounded subsets of

The compounds weak-∗ and strong-∗ are to be read «weak *star*» and «strong *star*»

$\mathfrak{X}$, the set $\mathfrak{S}$ of all **finite subsets** of $\mathfrak{X}$. This topology is usually denoted by $\sigma(\mathfrak{X}', \mathfrak{X})$. When $\mathfrak{X}'$ is endowed with this topology, we denote it by $\mathfrak{X}'_\sigma$, and we say that the couple $(\mathfrak{X}', \sigma(\mathfrak{X}', \mathfrak{X}))$ is the **weak dual** of $\mathfrak{X}$.

**Note that** $V$ is a neighborhood of the origin in $\mathfrak{X}'_\sigma$ if, and only if, $V$ contains the polar of a (non-empty) finite subset of $\mathfrak{X}$, that is, if there exist a finite subset $\{x_1, ..., x_n\} \subseteq \mathfrak{X}$ and $\rho \in \mathbb{R}^+$ such that $V \supseteq \{x' \in \mathfrak{X}' :: \sup_{i \in \mathbb{N}_n} |\langle x', x_i \rangle| \leqslant \rho\}$.

**7.25. Definition.** We call **strong dual topology** (or **strong-∗ topology**) on $\mathfrak{X}'$ the (locally convex) $\mathfrak{S}$-topology on $\mathfrak{X}'$ defined by taking as filtered by inclusion family of (non-empty) bounded subsets of $\mathfrak{X}$, the set $\mathfrak{S}$ consisting of all **nonempty and bounded subsets** of $\mathfrak{X}$. This topology is usually denoted by $b(\mathfrak{X}', \mathfrak{X})$. When $\mathfrak{X}'$ is endowed with this topology, we denote it by $\mathfrak{X}'_b$, and we say that the couple $(\mathfrak{X}', b(\mathfrak{X}', \mathfrak{X}))$ is the **strong dual** of $\mathfrak{X}$.

**Note that** $V$ is a neighborhood of the origin in $\mathfrak{X}'_b$ if, and only if, $V$ contains the polar of a (non-empty) bounded subset of $\mathfrak{X}$, that is if there exist a bounded subset $S \subseteq \mathfrak{X}$ and $\rho \in \mathbb{R}^+$ such that $V \supseteq \{x' \in \mathfrak{X}' :: \sup_{x \in S} |\langle x', x \rangle| \leqslant \rho\}$.

The next proposition collects some consequences of the definition.

**7.26. Proposition.** *Let $\mathfrak{X}$ be a topological vector space. The following assertions hold:*

***i.*** *Both the weak-∗ and the strong-∗ topologies are (Hausdorff) separated.*

***ii.*** *A generalized sequence $(x'_\lambda)_{\lambda \in \Lambda}$ converges to $x' \in \mathfrak{X}'$ for the weak-∗ topology if, and only if, it converges* **pointwise***, that is* **for every** *$x \in \mathfrak{X}$, the generalized sequence (of real numbers) $(\langle x'_\lambda, x \rangle)_{\lambda \in \Lambda}$ converges to $\langle x', x \rangle$.*

***iii.*** *A generalized sequence $(x'_\lambda)_{\lambda \in \Lambda}$ in $\mathfrak{X}'$ converges to $x' \in \mathfrak{X}'$ for the strong-∗ topology if, and only if, it converges to $x'$* **uniformly** *on every non-empty bounded subset of $\mathfrak{X}$, that is, if*

$$\sup_{x \in S} |\langle x'_\lambda - x', x \rangle| \to 0$$

*for every bounded (and non-empty) subset $S$ of $\mathfrak{X}$.*

***iv.*** *The strong-∗ topology is* **finer** *(***stronger***) than the weak-∗ topology.*

**Proof.** Statement ***i.*** follows from **Proposition 7.23.*a*)**. Statements ***ii.*** and ***iii.*** follow from **Proposition 7.23.*b*)**. Statement ***iv.*** is a consequence of ***ii.*** and ***iii.*** ■

## 7.4 The transpose operator

**Notation.** Given two **sets** $X, Y$, we denote by $\mathscr{F}(X, Y)$ the set of all maps from $X$ into $Y$. If $X$, $Y$ are **vector spaces**, we denote by $L(X, Y)$ the vector space of all **linear** maps from $X$ into $Y$. If $\mathfrak{X}, \mathfrak{Y}$ are **topological vector spaces**, we denote by $\mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ the set of **linear** and **continuous** maps from $\mathfrak{X}$ into $\mathfrak{Y}$.

**7.27. Definition.** Let $\mathfrak{X}, \mathfrak{Y}$ be topological vector spaces and $\mathfrak{X}', \mathfrak{Y}'$ their **continuous dual** spaces. Let $f: \mathfrak{X} \to \mathfrak{Y}$ be a **linear** and **continuous** map: $f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$. We call **transpose (map) of** $f$ the map

$f^\top \in \mathscr{F}(\mathfrak{Y}', \mathfrak{X}')$ defined by

$$y' \in \mathfrak{Y}' \mapsto f^\top(y') := y' \circ f \in \mathfrak{X}' \quad \textbf{for any } y' \in \mathfrak{Y}'.$$

The map $y' \circ f \in \mathfrak{X}' : x \in \mathfrak{X} \mapsto y'(f(x)) \in \mathbb{R}$ is called the **linear pull-back** of $y'$ through $f$.

**7.28. Remark.** Think about why for any given $f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ one has $f^\top \in \mathscr{F}(\mathfrak{Y}', \mathfrak{X}')$. Indeed, given $f$, its transpose $f^\top$ is the map

$$y \in \mathfrak{Y}' \mapsto \mathfrak{X}' \ni f^\top(y') := x \in \mathfrak{X} \mapsto \langle f^\top(y'), x \rangle := \langle y', f(x) \rangle \in \mathbb{R}.$$

The fact that the map $x \in \mathfrak{X} \mapsto \langle f^\top(y'), x \rangle := \langle y', f(x) \rangle \in \mathbb{R}$ is linear and continuous, i.e., that $f^\top$ has $\mathfrak{X}'$ as codomain, follows from the fact that $f^\top(y') := y' \circ f$ is the composition of two linear and continuous maps: $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{y'} \mathbb{R}$.

Note that, by definition,

$$\langle f^\top(y'), x \rangle := \langle y', f(x) \rangle \quad \forall x \in \mathfrak{X}, \ \forall y' \in \mathfrak{Y}'.$$

Summarizing, if $f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ then $f^\top \in \mathscr{F}(\mathfrak{Y}', \mathfrak{X}')$. Up to now, we only know the domain and codomain of the transpose of $f$. But we still don't know if given $f$, its transpose is linear and/or continuous.

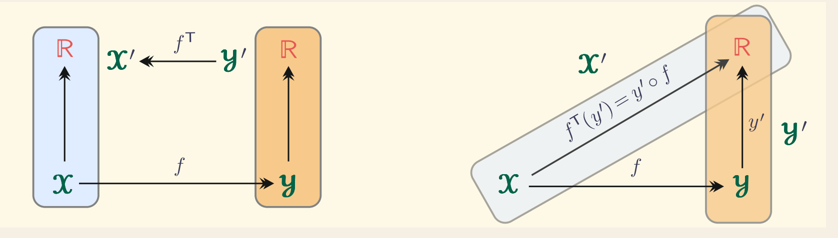



**Figure 7.2.** Given $f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$, we call **transpose of** $f$ the map $f^\top$ defined by $f^\top(y') := y' \circ f$ for any $y' \in \mathfrak{Y}'$.

### 7.4.1. Algebraic properties of the transpose operator

Before proving some (mainly purely) algebraic properties of the transpose operator, let us think about the concept we have introduced. The notion of transposition gives rise to three maps:

1. Given $f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$, for every $y' \in \mathfrak{Y}'$ the transpose map $f^\top$ defines the **linear** functional $f^\top(y')$ defined in $\mathfrak{X}'$ and which acts as follows

$$\langle f^\top(y'), x \rangle := \langle y', f(x) \rangle \quad \forall x \in \mathfrak{X}.$$

   The linear functional $f^\top(y')$ is called the **linear pull-back of** $y'$ **through** $f$.

2. Given $f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$, we can consider the map (which we are going to prove to be linear) $f^\top \in \mathscr{F}(\mathfrak{Y}', \mathfrak{X}')$ from $\mathfrak{Y}'$ to $\mathfrak{X}'$ which gives rise to the duality identity

$$\langle f^\top(y'), x \rangle = \langle y', f(x) \rangle \quad \forall x \in \mathfrak{X}, \ \forall y' \in \mathfrak{Y}'.$$

   The map $f^\top$ is called the **transpose map of** $f$. The transpose map of $f$ associates to every linear and continuous functional $y \in \mathfrak{Y}'$ its linear pull-back through $f$.

3. The **transposition operator** $^\top: f \to f^\top$ is a map from $\mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ to $\mathscr{F}(\mathfrak{Y}', \mathfrak{X}')$ and satisfies the relation

$$\langle f^\top(y'), x\rangle = \langle y', f(x)\rangle \quad \forall f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y}), \forall y' \in \mathfrak{Y}', \forall x \in \mathfrak{X}.$$

Concerning these three maps we have the following result.

**7.29. Proposition.** *Let $\mathfrak{X}, \mathfrak{Y}$ be topological vector spaces and $\mathfrak{X}', \mathfrak{Y}'$ their* **continuous dual** *spaces. Let $f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ and $f^\top \in \mathscr{F}(\mathfrak{Y}', \mathfrak{X}')$ be the transpose of $f$. The following assertions hold:*

*i. The map $f^\top: \mathfrak{Y}' \to \mathfrak{X}'$ is linear. In other words, the transpose map* **of** *any linear and continuous map is linear. In symbols: $f^\top \in L(\mathfrak{Y}', \mathfrak{X}')$.*

*ii. If $j_{\mathfrak{X}}$ and $j_{\mathfrak{X}'}$ are, respectively, the identity maps in $\mathfrak{X}$ and $\mathfrak{X}'$, then*

$$(j_{\mathfrak{X}})^\top \equiv j_{\mathfrak{X}'}.$$

*iii. If $f_1, f_2 \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ and $\lambda_1, \lambda_2 \in \mathbb{K}$, then $(\lambda_1 f_1 + \lambda_2 f_2)^\top = \lambda_1 f_1^\top + \lambda_2 f_2^\top$. In other words, the transposition operator is linear: $^\top \in L(\mathscr{L}(\mathfrak{X}, \mathfrak{Y}), L(\mathfrak{Y}', \mathfrak{X}'))$.*

*iv. Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ be three topological vector spaces and let $f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ and $g \in \mathscr{L}(\mathfrak{Y}, \mathfrak{Z})$. We then have*

$$(g \circ f)^\top = f^\top \circ g^\top.$$

**7.30. Remark.** Note that in *i.* we just stated that $f^\top: \mathfrak{Y}' \to \mathfrak{X}'$ is linear; no assertion has been made about the continúity of $f^\top$. Indeed, in general, the continúity of the transpose of $f$ depends on the topologies of $\mathfrak{X}$ and $\mathfrak{Y}$. The continúity of $f^\top$ is guaranteed when (cf. Proposition 7.35) both $\mathfrak{Y}'$ and $\mathfrak{X}'$ are endowed with the **weak dual topology** (and in this case $f^\top \in \mathscr{L}(\mathfrak{Y}'_\sigma, \mathfrak{X}'_\sigma)$) **or** when $\mathfrak{Y}'$ and $\mathfrak{X}'$ are endowed with the **strong dual topology** (and in this case $f^\top \in \mathscr{L}(\mathfrak{Y}'_b, \mathfrak{X}'_b)$).

**PROOF.** *i.* For every $y_1', y_2' \in \mathfrak{Y}'$ and any $\lambda_1, \lambda_2 \in \mathbb{K}$ one has

$$\begin{aligned} f^\top(\lambda_1 y_1' + \lambda_2 y_2') &= (\lambda_1 y_1' + \lambda_2 y_2') \circ f \\ &= \lambda_1 (y_1' \circ f) + \lambda_2 (y_2' \circ f) \\ &= \lambda_1 f^\top(y_1') + \lambda_2 f^\top(y_2'). \end{aligned}$$

*ii.* It is trivial, but let us prove this to practice. The identity map $j: \mathfrak{X} \to \mathfrak{X}$ has for transpose the identity map $(j_{\mathfrak{X}})^\top: \mathfrak{X}' \to \mathfrak{X}'$. Indeed

$$(j_{\mathfrak{X}})^\top(x') = x' \circ j_{\mathfrak{X}} = x' = j_{\mathfrak{X}'}(x').$$

The arbitrariness of $x' \in \mathfrak{X}'$ proves the assertion.

*iii.* If $f_1, f_2 \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ and $\lambda_1, \lambda_2 \in \mathbb{K}$, then

$$\begin{aligned} (\lambda_1 f_1 + \lambda_2 f_2)^\top(y') &= y' \circ (\lambda_1 f_1 + \lambda_2 f_2) \\ &= \lambda_1 y \circ (f_1) + \lambda_2 y(f_2) \\ &= \lambda_1 f_1^\top(y') + \lambda_2 f_2^\top(y'). \end{aligned}$$

The arbitrariness of $y' \in \mathfrak{Y}'$ proves the assertion.

*iv.* First note that $g \circ f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Z})$ and $g^\top(z') = z' \circ g \in \mathfrak{Y}'$. Hence,

$$(g \circ f)^\top(z') = z' \circ (g \circ f)$$

$$\begin{aligned} &= (z'\circ g)\circ f \\ &= g^\top(z')\circ f \\ &= f^\top(g^\top(z')) \\ &= (f^\top\circ g^\top)(z'). \end{aligned}$$

The arbitrariness of $z'\in\mathfrak{Z}'$ proves the assertion. ■■■■

#### 7.4.1.1. Relations between the kernel of $f$ and the range of $f^\top$.

**7.31. Proposition.** *For any linear and continuous map $f\in\mathscr{L}(\mathfrak{X},\mathfrak{Y})$, the kernel of the linear map $f^\top$ coincides with the orthogonal of the image space of $f$. In other words:*

$$\ker(f^\top) = (\mathrm{Im}\, f)^\circ = (\mathrm{Im}\, f)^\perp \quad \forall f\in\mathscr{L}(\mathfrak{X},\mathfrak{Y}).$$

**7.32. Remark.** The proof relies on **Proposition 7.15.*a*)**. Let us recall the result. If $A$ is a vector **sub**space of $\mathfrak{X}$ then $A^\circ$ coincides with the **orthogonal of** $A$, id est, with the subspace $A^\perp := \{x'\in \mathfrak{X}' :: \langle x', x\rangle = 0 \text{ for every } x\in A\}$.

**Proof.** The key point is that $\mathrm{Im}\, f = f(\mathfrak{X})$ is a vector subspace of $\mathfrak{Y}$. Therefore, due to **Proposition 7.15.*b*)**,One has

$$\begin{aligned} \mathrm{Ker}(f^\top) &= \{y'\in\mathfrak{Y}' :: f^\top(y') = 0\} \\ &= \{y'\in\mathfrak{Y}' :: \langle f^\top(y'), x\rangle = 0 \text{ for all } x\in\mathfrak{X}\} \\ &= \{y'\in\mathfrak{Y}' :: \langle y', f(x)\rangle = 0 \text{ for all } x\in\mathfrak{X}\} \\ &= \{y'\in\mathfrak{Y}' :: \langle y', y\rangle = 0 \text{ for all } y\in f(\mathfrak{X})\} \\ &= f(\mathfrak{X})^\perp \\ &= (\mathrm{Im}\, f)^\circ. \end{aligned}$$

This concludes the proof. ■■■■

**7.33. Corollary. (On the Injectivity of the transpose of $f$)** *Let $\mathfrak{X},\mathfrak{Y}$ be topological vector spaces and $\mathfrak{X}',\mathfrak{Y}'$ their* **continuous dual** *spaces. Let $f\in\mathscr{L}(\mathfrak{X},\mathfrak{Y})$ and $f^\top$ be the transpose of $f$. If the image space $f(\mathfrak{X})$ is dense in $\mathfrak{Y}$ (in particular, if $f$ is surjective) then the transpose $f^\top:\mathfrak{Y}'\to\mathfrak{X}'$ of $f$ is injective.*

**7.34. Remark.** The injectivity of $f^\top$ means that for any $y_1', y_2'\in\mathfrak{Y}'$, if $\langle y_1', f(x)\rangle = \langle y_2', f(x)\rangle$ for any $x\in\mathfrak{X}$ then $y_1'\equiv y_2'$.

**Proof. Step 1**. It is sufficient to prove that if $\mathfrak{M}$ is a **dense** linear subspace of $\mathfrak{Y}$, then $\mathfrak{M}^\circ = \mathfrak{M}^\perp = \{0\}$. Indeed, after that, setting $\mathfrak{M} = \mathrm{Im}\, f$, we can infer that

$$\ker(f^\top) = (\mathrm{Im}\, f)^\circ = \{0\},$$

i.e., that $f$ is injective. Therefore, let us show that if $\mathfrak{M}$ is a **dense** linear subspace of $\mathfrak{Y}$, then $\mathfrak{M}^\circ = \mathfrak{M}^\perp = \{0\}$. Let $y'\in\mathfrak{M}^\circ$. Then, $\langle y', m\rangle = 0$ for every $m\in\mathfrak{M}$ (cf. **Proposition 7.15.*b*)**). Since $y'$ is a **continuous** and **linear** functional from $\mathfrak{Y}$ into $\mathbb{K}$ and since the topology of $\mathbb{K}$ is (Hausdorff) *separated*, the principle of extension of the identities shows that $\langle y', y\rangle = 0$ for every $y\in\bar{\mathfrak{M}} = \mathfrak{Y}$. Hence $\mathfrak{M}^\circ = \mathfrak{M}^\perp = \{y'\in\mathfrak{Y}' :: \langle y', y\rangle = 0 \text{ for every } y\in\mathfrak{Y}\} = \{0\}$. This completes the proof. ■■■■

### 7.4.2. Some topological property of the Transposed map

**7.35. Proposition. (Continuity of the transpose map)** *Let $\mathfrak{X}, \mathfrak{Y}$ be topological vector spaces and $\mathfrak{X}', \mathfrak{Y}'$ their* **continuous dual** *spaces. Let $f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ and $f^\top$ be the transpose of $f$. We already observed that $f^\top$ is linear. In addition, $f^\top$ is* **continuous** *in both of the following two cases:*

- *When $\mathfrak{Y}', \mathfrak{X}'$ are endowed with the* **weak dual topology***: $f^\top \in \mathscr{L}(\mathfrak{Y}'_\sigma, \mathfrak{X}'_\sigma)$.*
- *When $\mathfrak{Y}', \mathfrak{X}'$ are endowed with the* **strong dual topology***: $f^\top \in \mathscr{L}(\mathfrak{Y}'_b, \mathfrak{X}'_b)$.*

**7.36. Remark.** Of course, since $\sigma(\mathfrak{X}', \mathfrak{X})$ and $\sigma(\mathfrak{Y}', \mathfrak{Y})$ are (respectively) coarser then $b(\mathfrak{X}', \mathfrak{X})$ and $b(\mathfrak{Y}', \mathfrak{Y})$, and the continúity is preserved by coarsening the topology of the codomain and/or by refining the topology of the domain, we also have

$$f^\top \in \mathscr{L}(\mathfrak{Y}'_b, \mathfrak{X}'_\sigma).$$

Summarizing, we have that $f^\top \in \mathscr{L}(\mathfrak{Y}'_\sigma, \mathfrak{X}'_\sigma) \cap \mathscr{L}(\mathfrak{Y}'_b, \mathfrak{X}'_\sigma) \cap \mathscr{L}(\mathfrak{Y}'_b, \mathfrak{X}'_b)$.

**Proof.**

**[$f^\top \in \mathscr{L}(\mathfrak{Y}'_\sigma, \mathfrak{X}'_\sigma)$]** Let both $\mathfrak{X}'$ and $\mathfrak{Y}'$ be endowed with their **weak dual** topologies. We have to show that for any generalized sequence $(y'_\lambda)_{\lambda\in\Lambda}$ in $\mathfrak{Y}'$, the following implication holds:

$$(y'_\lambda)_{\lambda\in\Lambda} \to 0 \quad \text{in } \mathfrak{Y}'_\sigma \quad \Rightarrow \quad (f^\top(y'_\lambda))_{\lambda\in\Lambda} \to 0 \quad \text{in } \mathfrak{X}'_\sigma.$$

Let $(y'_\lambda)_{\lambda\in\Lambda}$ be a generalized sequence converging to zero in $\mathfrak{Y}'_\sigma$. Then, for any $x \in \mathfrak{X}$, the generalized sequence $(\langle y'_\lambda, f(x)\rangle)_{\lambda\in\Lambda}$ converges to zero (cf. Proposition 7.26). It follows that the generalized sequence $(\langle f^\top(y'_\lambda), x\rangle)_{\lambda\in\Lambda}$ converges to zero, and this shows that the generalized sequence $(f^\top(y'_\lambda))_{\lambda\in\Lambda}$ converges to zero in $\mathfrak{X}'_\sigma$.

**[$f^\top \in \mathscr{L}(\mathfrak{Y}'_b, \mathfrak{X}'_b)$]** Let both $\mathfrak{X}'$ and $\mathfrak{Y}'$ be endowed with their **strong dual** topologies. Let $A$ be a bounded subset of $\mathfrak{X}$, and let $B = f(A)$ the image of $A$ under $f$. Since $f$ is continuous and homogeneous, $B$ is a bounded subset of $\mathfrak{Y}$ (cf. Proposition 3.48). Thus

$$\begin{aligned}
\mathfrak{p}_{A^\circ}(f^\top(y')) &= \sup_{x\in A} |\langle f^\top(y'_\lambda), x\rangle| \\
&= \sup_{x\in A} |\langle y'_\lambda, f(x)\rangle| \\
&= \sup_{y\in B} |\langle y'_\lambda, y\rangle| \\
&= \mathfrak{p}_{B^\circ}(y').
\end{aligned}$$

Proposition 4.37 shows the continúity of $f^\top$.

**Recall Proposition 4.37**: Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two **locally convex** spaces and $T$ a **linear** map from $\mathfrak{X}$ into $\mathfrak{Y}$. **Assume** that $\mathfrak{P}$ is a **basis** of continuous seminorms on $\mathfrak{X}$ and $\mathfrak{Q}$ is a **basis** of continuous seminorms on $\mathfrak{Y}$. **Then** $T: \mathfrak{X} \to \mathfrak{Y}$ is continuous if, and only if, for every continuous seminorm $\mathfrak{q} \in \mathfrak{Q}$, there exists a continuous seminorm $\mathfrak{p} \in \mathfrak{P}$ and a constant $c_\mathfrak{q} > 0$ such that $|Tx|_\mathfrak{q} \leqslant c_\mathfrak{q} |x|_\mathfrak{p}$ $\forall x \in \mathfrak{X}$. Note that here both $c_\mathfrak{q}$ and $\mathfrak{p}$ depend on $\mathfrak{q}$ but not on $x$.

### 7.4.3. Transposed of a topological isomorphism

Let us recall that given two topological vector spaces $\mathfrak{X}, \mathfrak{Y}$, we say that $f \in \mathscr{L}(\mathfrak{X}, \mathfrak{Y})$ is a topological isomorphism if $f$ is both an **isomorphism of vector spaces** and a **homeomorphism of topological spaces**.

**7.37. Proposition.** *Let $f$ be a* **topological isomorphism** *of $\mathfrak{X}$ onto $\mathfrak{Y}$. Then the transpose map $f^\top$ is a topological isomorphism both of $\mathfrak{Y}'_\sigma$ onto $\mathfrak{X}'_\sigma$ and of $\mathfrak{Y}'_b$ onto $\mathfrak{X}'_b$. Moreover*

$$f^{-\top} := (f^{-1})^\top \equiv (f^\top)^{-1},$$

*i.e., the inverse of the transpose if equal to the transpose of the inverse.*

**PROOF.** To easy the notation, we set $g = f^{-1}$. By hypothesis, $g$ is linear and continuous of $\mathfrak{Y}$ into $\mathfrak{X}$. Therefore $g^\top$ exists and moreover

$$f \circ g = j_{\mathfrak{Y}}, \quad g \circ f = j_{\mathfrak{X}}.$$

Hence, thanks to **Proposition 7.29** we get

$$j_{\mathfrak{Y}'} = (j_{\mathfrak{Y}})^\top = g^\top \circ f^\top \quad \textbf{and} \quad j_{\mathfrak{X}'} = (j_{\mathfrak{X}})^\top = f^\top \circ g^\top.$$

This shows that $g^\top = (f^\top)^{-1}$, i.e., $(f^{-1})^\top = (f^\top)^{-1}$. In particular, since $(f^{-1})^\top$ is continuous, so is $(f^\top)^{-1}$. Thus, $f^\top$ is a topological isomorphism both of $\mathfrak{Y}'_\sigma$ on $\mathfrak{X}'_\sigma$ and of $\mathfrak{Y}'_b$ on $\mathfrak{X}'_b$. ■

## 7.5 | Canonical injection among dual spaces

**7.38. Theorem.** *Let $\mathfrak{M} := (M, \tau_M)$, $\mathfrak{X} := (X, \tau)$ be two topological vector spaces such that $\mathfrak{M} \hookrightarrow \mathfrak{X}$ and $\mathfrak{X} = \mathsf{Cl}_{\mathfrak{X}}(M)$. In other words,* **suppose** *that*

- *$M \trianglelefteq X$;*
- *the* **canonical injection** *of $\mathfrak{M}$ into $\mathfrak{X}$, given by $j : x \in \mathfrak{M} \mapsto x \in \mathfrak{X}$, is continuous;*
- *the injection $j$ has a* **dense image**.

$\mathsf{Cl}_{\mathfrak{X}}(M)$ means that the closure of $M$ is taken in the topology of $\mathfrak{X}$

**Then** *the following two assertions hold:*

*i.* *The transpose $j^\top$ of $j$* **continuously inject** *$\mathfrak{X}'$ into $\mathfrak{M}'$.*

*ii.* *Every linear* **form** *$u \in M^*$ which is continuous on $(M, \tau \mid M)$, that is, continuous with respect to the subspace topology induced on $M$ by $\mathfrak{X}$, is extendable (uniquely) to a continuous linear functional $u^\#$ on $\mathfrak{X}$ and $j^\top(u^\#) = u$.*

**PROOF.** The statement ***i.*** follows from **Corollary 7.33** and **Proposition 7.35**. Existence and uniqueness of the extension $u^\#$ is a consequence of the **principle of extension by continuity** (cf. **Theorem 3.36**), because $\mathbb{K}$ is Hausdorff separated and complete. Finally, since form the algebraic point of view $j$ coincides with the *restriction* operator, we have $u = u^\# \circ j$, and the last equality follows. ■

**7.39. Remark.** Often, since the transpose $j^\top$ of $j$ inject $\mathfrak{X}'$ into $\mathfrak{M}'$, one identifies $\mathfrak{X}'$ with a vector subspace of $\mathfrak{M}'$. The previous theorem than states not only that $\mathfrak{X}' \trianglelefteq \mathfrak{M}'$ but even

$$\mathfrak{X}' \hookrightarrow \mathfrak{M}'.$$

Moreover, a sufficient condition for an element on $u \in \mathfrak{M}'$ to be extendable to an element of $\mathfrak{X}'$ is that $u$ is continuous on $(M, \tau \mid M)$, that is, continuous with respect to the subspace topology induced on $M$ by $\mathfrak{X}$.

**Example 7.40.** Let $\Omega$ be an open subset of $\mathbb{R}^N$. Let us particularize the injection theorem to the case $\mathfrak{M}:=\mathfrak{D}(\Omega)$ and $\mathfrak{X}:=\mathfrak{K}(\Omega)$. Clearly, $C_c^\infty(\Omega)\trianglelefteq C_c(\Omega)$ and, moreover, the topology of $\mathfrak{D}(\Omega)=(C_c^\infty(\Omega),\tau_{\mathrm{LF}}^\infty)$ is finer than the subspace topology induced on $C_c^\infty(\Omega)$ by $\mathfrak{K}(\Omega)=(C_c(\Omega),\tau_{\mathrm{LF}})$. Therefore, the canonical injection $j:\mathfrak{D}(\Omega)\hookrightarrow\mathfrak{K}(\Omega)$ is continuous. Also, **Theorem 6.61** shows that $C_c^h(\Omega)=j(C_c^h(\Omega))$ is dense in $\mathfrak{K}(\Omega)$. According to the injection theorem, the transpose map $j^\top$ of $j$ injects $\mathfrak{K}'(\Omega)$ into $\mathfrak{D}'(\Omega)$ so that one can identify $\mathfrak{K}'(\Omega)$ with a vector subspace of $\mathfrak{D}'(\Omega)$ and one can also write $\mathfrak{K}'(\Omega)\trianglelefteq\mathfrak{D}'(\Omega)$ or even $\mathfrak{K}'(\Omega)\hookrightarrow\mathfrak{D}'(\Omega)$. Moreover, a sufficient condition for a distribution $u\in\mathfrak{D}'(\Omega)$ to be extendable to a Radon measure, is that $u$ is continuous on $(C_c^\infty(\Omega),\tau_{\mathrm{LF}}\,|\,C_c^\infty(\Omega))$, that is, continuous with respect to the subspace topology induced on $C_c^\infty(\Omega)$ by $\mathfrak{K}(\Omega)$.

Recall that a Radon measure is, by definition, an element of $\mathfrak{K}'(\Omega)$

**7.41. Theorem. (Consistency)** *Let $(\mathfrak{M},\mathfrak{X})$ and $(\mathfrak{N},\mathfrak{Y})$ be two pairs of topological vector spaces such that*

$$\begin{aligned}\mathfrak{M}&\hookrightarrow\mathfrak{X}\quad\text{and}\quad \mathsf{Cl}_{\mathfrak{X}}(M)=\mathfrak{X},\\ \mathfrak{N}&\hookrightarrow\mathfrak{Y}\quad\text{and}\quad \mathsf{Cl}_{\mathfrak{Y}}(N)=\mathfrak{Y}.\end{aligned}$$

*We identify $\mathfrak{X}'$ to a subspace of $\mathfrak{M}'$ and $\mathfrak{Y}'$ to a subspace of $\mathfrak{N}'$:*

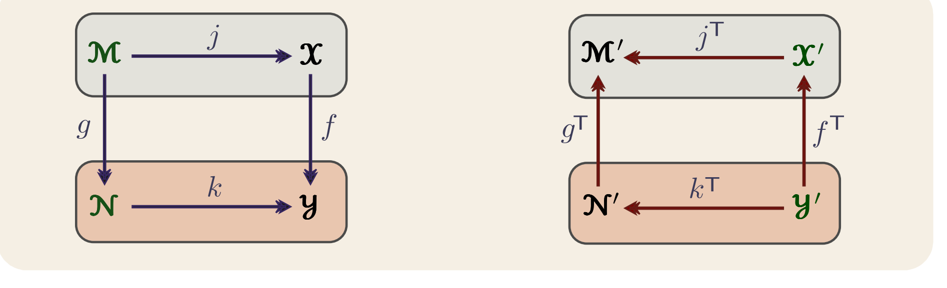
$$\begin{aligned}\mathfrak{X}'&\hookrightarrow\mathfrak{M}',\\ \mathfrak{Y}'&\hookrightarrow\mathfrak{N}'.\end{aligned}$$

*Let $f,g$ be* **linear** *and* **continuous** *maps from $\mathfrak{M}$ to $\mathfrak{N}$ and from $\mathfrak{X}$ to $\mathfrak{Y}$:*

$$f:\mathfrak{X}\to\mathfrak{Y},\quad g:\mathfrak{M}\to\mathfrak{N}.$$

*If $g$ is the restriction of $f$ to $\mathfrak{M}$, then $f^\top:\mathfrak{Y}'\to\mathfrak{X}'$ is the restriction of $g^\top:\mathfrak{N}'\to\mathfrak{M}'$ to $\mathfrak{Y}'$.*

**Proof.** Let us denote by $j$ the canonical injection of $\mathfrak{M}$ into $\mathfrak{X}$. Also, denote by $k$ the canonical injection of $\mathfrak{N}$ into $\mathfrak{Y}$. Stating that $g$ is the restriction of $f$ is equivalent to say $k\circ g=f\circ j$. By transposition, in agreement with **Proposition 7.29.*iv***, we obtain that $g^\top\circ k^\top=j^\top\circ f^\top$. In terms of **commutative** diagrams we have:




Identifying $\mathfrak{X}'$ (resp. $\mathfrak{Y}'$) with a subspace of $\mathfrak{M}'$ (resp. $\mathfrak{N}'$), the relation $g^\top\circ k^\top=j^\top\circ f^\top$ expresses that $f^\top$ coincides with the restriction of $g^\top$ to $\mathfrak{Y}'$. ■

# 8

## Spaces of Distributions

### 8.1 Distributions on open subsets of $\mathbb{R}^N$

Recall that $\mathcal{D}(\Omega) := (C_c^\infty(\Omega), \tau_{\mathrm{LF}})$ denotes the complete locally convex (topological vector) space $C_c^\infty(\Omega)$ endowed with the topology $\tau_{\mathrm{LF}}$ inductive limit of FRECHET spaces.

**8.1. Definition.** Let $\Omega$ be an open subset (bounded or not) of $\mathbb{R}^N$, with $N \geqslant 1$. We call **distribution** on $\Omega$ every linear and continuous functional on $\mathcal{D}(\Omega)$. The space of distributions on $\Omega$ is nothing but the **dual space** of $\mathcal{D}(\Omega)$ and, therefore, it is denoted by $\mathcal{D}'(\Omega)$. In other words, by definition,

Continuous with respect to the topology $\tau_{\mathrm{LF}}$ of $\mathcal{D}(\Omega)$

$$\mathcal{D}'(\Omega) := (\mathcal{D}(\Omega))'. \tag{8.1}$$

Let $T \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. The value of $T$ at $\varphi$ will be denoted by $T(\varphi)$ or $\langle T, \varphi \rangle$. ⤛

**8.2. Remark.** Other ways to denote the value of $T$ at $\varphi$ are

$$\int_\Omega \varphi(x)\,\mathrm{d}T(x) \quad \textbf{and} \quad \int_\Omega T(x)\varphi(x)\,\mathrm{d}x. \tag{8.2}$$

However, the previous symbols are more common when $T \in \mathcal{M}(\Omega) := \mathcal{K}'(\Omega)$, i.e., when $T$ is a Radon measure on $\Omega$. ⋯

**Algebraic dual of $C_c^\infty(\Omega)$.** The set $\mathcal{D}'(\Omega)$ can be structured as a vector space in the usual natural way. For every $T, T_1, T_2 \in \mathcal{D}'(\Omega)$, $\varphi \in \mathcal{D}(\Omega)$ and $\lambda \in \mathbb{C}$ we set

$$\langle T_1 + T_2, \varphi \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle \quad \textbf{and} \quad \langle \lambda T, \varphi \rangle = \lambda \langle T, \varphi \rangle.$$

The natural algebraic structure on $\mathcal{D}'(\Omega)$ is the one induced by the **algebraic dual** $(C_c^\infty(\Omega))^*$.

**Topologies on $\mathcal{D}'(\Omega)$.** The vector space $\mathcal{D}'(\Omega)$, being the continuous dual of a locally convex space, can be naturally endowed with both the strong-dual topology and the weak-dual topology. Both these topologies turn $\mathcal{D}'(\Omega)$ into a Hausdorff separated locally convex space. In principle, according to the general notation introduced in Definition 7.24 and Definition 7.25, one has to denote by $\mathcal{D}'(\Omega)_\sigma$ and $\mathcal{D}'(\Omega)_b$, respectively, the locally convex space of distributions endowed with the weak-dual topology and with the strong-dual topology. However, the context will clarify under which topology a specific result holds.

- The **strong-dual topology** (cf. Definition 7.25), corresponds to the uniform convergence over bounded subsets of $\mathcal{D}(\Omega)$. More explicitly, from the characterization of bounded subsets of $\mathcal{D}(\Omega)$ (cf. Proposition 6.58 with $k = \infty$), this means that a **generalized sequence** (a **net**)

$(T_\lambda)_{\lambda\in\Lambda}$ converges to $T\in\boldsymbol{\mathcal{D}}'(\Omega)$, **if, and only if**, for any $K\in\mathfrak{K}_\Omega$ and any bounded subset $S$ of $\boldsymbol{\mathcal{D}}_K(\Omega)$ we have

$$\mathfrak{p}_S(T_\lambda - T)\equiv\sup_{\varphi\in S}|\langle T-T_\lambda,\varphi\rangle|\to 0 \quad \text{in } \mathbb{R}.$$

Recall that, $S$ is bounded in $\boldsymbol{\mathcal{D}}_K(\Omega)$ if, and only if,

$$\left(\operatorname{supp}_\Omega\varphi\subseteq K\ \ \forall\varphi\in S\right)\quad\textbf{and}\quad\left(\sup_{\varphi\in S}\mathfrak{p}_{K,m}(\varphi)<\infty\quad\forall m\in\mathbb{N}\right),$$

with $\mathfrak{p}_{K,m}(\varphi)=\sup_{|\alpha|\leqslant m}\sup_{x\in K}|\partial^\alpha\varphi|$. Indeed, recall that $\boldsymbol{\mathcal{D}}_K(\Omega)$ is a Frechét subspace of $\boldsymbol{\mathcal{E}}(\Omega)$ (cf. **Proposition 6.54**).

However, more often, we will endow $\boldsymbol{\mathcal{D}}'(\Omega)$ with

- The **weak-dual topology** (cf. **Definition 7.24**), corresponds to the uniform convergence over finite (in cardinality) subsets of $\boldsymbol{\mathcal{D}}(\Omega)$. More explicitly, first, recall that (cf. **Remark 7.22**) the weak-dual topology on $\boldsymbol{\mathcal{D}}'(\Omega)$ is defined by the **filtering** family of seminorms $\{\mathfrak{p}_S\colon\boldsymbol{\mathcal{D}}'(\Omega)\to\mathbb{R}_+\}_{S\in\mathfrak{S}_\sigma(\boldsymbol{\mathcal{D}}(\Omega))}$ with $\mathfrak{S}_\sigma(\boldsymbol{\mathcal{D}}(\Omega))$ consisting of all subsets of $\boldsymbol{\mathcal{D}}(\Omega)$ having finite cardinality and $\mathfrak{p}_S(T)=\sup_{\varphi\in S}|\langle T,\varphi\rangle|$. The family of seminorms $\{\mathfrak{p}_\varphi\colon\boldsymbol{\mathcal{D}}'(\Omega)\to\mathbb{R}_+\}_{\varphi\in\boldsymbol{\mathcal{D}}(\Omega)}$ defined for any $T\in\boldsymbol{\mathcal{D}}'(\Omega)$ by

  $$\mathfrak{p}_\varphi(T)=|\langle T,\varphi\rangle|,$$

  is a basis of continuous seminorms for $\boldsymbol{\mathcal{D}}'(\Omega)_\sigma$. Therefore, it generates the same locally convex topology of $\{\mathfrak{p}_\varphi\colon\boldsymbol{\mathcal{D}}'(\Omega)\to\mathbb{R}_+\}_{\varphi\in\boldsymbol{\mathcal{D}}(\Omega)}$. Thus, a **generalized sequence** (a **net**) $(T_\lambda)_{\lambda\in\Lambda}$ converges to $T\in\boldsymbol{\mathcal{D}}'(\Omega)$, for the weak-dual topology, **if, and only if**, for every $\varphi\in\boldsymbol{\mathcal{D}}(\Omega)$ the real-valued generalized sequence

  $$(\mathfrak{p}_\varphi(T_\lambda-T))_{\lambda\in\Lambda}=|\langle T-T_\lambda,\varphi\rangle|\to 0\quad\text{in }\mathbb{R}.$$

  In other words, the weak-dual topology is nothing but the topology of **(simple) pointwise convergence** on $\boldsymbol{\mathcal{D}}(\Omega)$.

  ○

The following characterization holds.

**8.3. Proposition.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$, and let $T$ be a* **linear form** *on $C_c^\infty(\Omega)$, i.e., an element of the* **algebraic dual** *$(C_c^\infty(\Omega))^*$. The following assertions are equivalent:*

***i.*** *$T$ is a distribution on $\Omega$. That is $T$ is in the* **continuous dual** *$\boldsymbol{\mathcal{D}}'(\Omega)$.*

***ii.*** *$T$ is sequentially continuous on $\boldsymbol{\mathcal{D}}(\Omega)$.*

***iii.*** *For every compact subset $K$ of $\Omega$, the restriction of $T$ to $\boldsymbol{\mathcal{D}}_K(\Omega)$ is (sequentially) continuous on $\boldsymbol{\mathcal{D}}_K(\Omega)$.*

***iv.*** *For every compact subset $K\in\mathfrak{K}_\Omega$ of $\Omega$ there exists a positive constant $c_K$ and a nonnegative integer $m_K\in\mathbb{N}$ such that $\mathfrak{p}_\varphi(T)\leqslant c_K\ \mathfrak{p}_{K,m_K}(\varphi)$ for every $\varphi\in C_K^\infty(\Omega)$. Here, $c_K$ and $m_K$ may depend on $K$, but not on $\varphi$. More explicitly, it must hold*

$$|\langle T,\varphi\rangle|\leqslant c_K\sup_{|\alpha|\leqslant m_K}\left(\sup_{x\in K}|\mathrm{D}^\alpha\varphi(x)|\right)\quad\textbf{for all }\varphi\in C_K^\infty(\Omega)\tag{8.3}$$

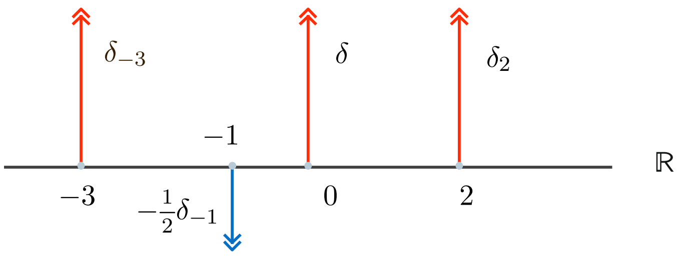



**Figure 8.1.** In $1d$ the Dirac delta distribution at $x_0 \in \mathbb{R}$ is schematically represented by an arrow based at $x_0$, of length one, and pointing upward. If $\lambda \in \mathbb{C}$, the Dirac delta distribution $\lambda\delta_{x_0}$ at $x_0 \in \mathbb{R}$ is schematically represented by an arrow based at $x_0$, of length $|\lambda|$, and pointing in the same direction of $\lambda \in \mathbb{C}$.

**Proof.** The equivalence of ***i.***,***ii.*** and ***iii.*** follows from the more general **Proposition 6.59** because $\mathbb{C}$ is a locally convex space. The equivalence of ***iii.*** and ***iv.*** follows from **Corollary 4.39** as soon as one recalls that the topology of $\mathcal{D}_K(\Omega)$ is defined by the family of seminorms $(\mathfrak{p}_{K,m})_{m \in \mathbb{N}}$. ■

**Example 8.4.** (Trivial) Let $\Omega \subseteq \mathbb{R}^N$ be an open set. The null functional

$$0: \varphi \in C_c^\infty(\Omega) \mapsto 0 \in \mathbb{C}$$

is a distribution on $\Omega$. This is trivial to show in a lot of different ways. If we want to use **Proposition 8.3.*iv.***, then we have to show that for every $K \in \mathfrak{K}_\Omega$ there exist $c_K > 0$ and $m_K \in \mathbb{N}$ such that

$$|\langle T, \varphi\rangle| = 0 \leqslant c_K \sup_{|\alpha| \leqslant m_K} \left(\sup_{x \in K} |\mathsf{D}^\alpha \varphi(x)|\right) \quad \textbf{for all } \varphi \in C_K^\infty(\Omega). \tag{8.4}$$

Clearly, every $c_K > 0$, as well as every $m_K \in \mathbb{N}$, does the job. For example, we can set $c_K = 1$ and $m_K = 0$. Note that these choices of $c_K$ and $m_K$ are valid regardless of the compact subset $K \in \mathfrak{K}_\Omega$. This is a rare circumstance. Instead, it is often the case that $m_K$ does not depend on $K \in \mathfrak{K}_\Omega$ although $c_K$ does. In this case, one says that the distribution is of finite order and, more precisely, that the distribution has order less than or equal to $m_K$ (cf. **Definition 8.8**). Thus, the fact that one can choose $m_K = 0$ regardless of $K$ tells us that the null functional is a distribution of order zero.

**Example 8.5.** (P. Dirac) Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $x_0 \in \Omega$. The linear form

$$\delta_{x_0}: \varphi \in C_c^\infty(\Omega, \mathbb{C}) \mapsto \varphi(x_0) \in \mathbb{C}$$

is a distribution on $\Omega$. Indeed, for any compact subset $K \in \mathfrak{K}_\Omega$ there exists a positive constant $c_K := 1$ and a nonnegative integer $m_K := 0 \in \mathbb{N}$ such that for all $\varphi \in C_K^\infty(\Omega, \mathbb{C})$

$$|\langle \delta_{x_0}, \varphi\rangle| \equiv |\varphi(x_0)| \leqslant c_K\, \mathfrak{p}_{K,m_K}(\varphi) \equiv \sup_{x \in K} |\varphi(x)|.$$

Indeed, if $x_0 \notin K$ then $\mathfrak{p}_\varphi(\delta_{x_0}) = 0$, while if $x_0 \in K$ then $|\varphi(x_0)| \leqslant \sup_{x \in K} |\varphi(x)|$. Note that both $c_K$ and $m_K$ do not depend on $K \in \mathfrak{K}_\Omega$. This is a rare circumstance. Instead, it is often the case that $m_K$ does not depend on $K \in \mathfrak{K}_\Omega$ although $c_K$ does. In this case, one says that the distribution is of finite order and, more precisely, that the distribution has order less than or equal to $m_K$ (cf. **Definition 8.8**). Note that when $m_K = 0$ one can replace the sentence «*of order less than or*

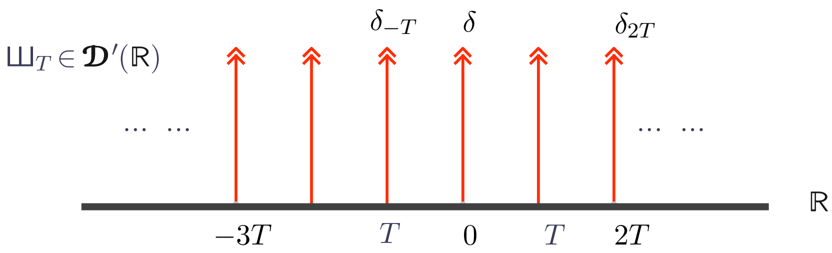



**Figure 8.2.** The Dirac **comb**, also known as an **impulse train**, **sampling function**, or as **Shah** function because its «graph» resembles the shape of the Cyrillic letter sha «$Ш_T$») is an infinite series of Dirac distributions spaced at intervals of $T$, for some given *period* $T$.

*equal zero*» with the sentence «*of order zero*». Thus, the Dirac delta distribution is a distribution of order zero.

The distribution $\delta_{x_0} \in \mathcal{D}'(\Omega)$ is called the **Dirac delta** (**distribution**) concentrated at $x_0$, or the **Dirac measure** concentrated at $x_0$ (see **Example 8.22**). If $0 \in \Omega$ the Dirac delta at $0$ is simply denoted by $\delta$. Clearly, if $\lambda \in \mathbb{C}$, then also $\lambda\delta_{x_0} \in \mathcal{D}'(\Omega)$ and we say that $\lambda\delta_{x_0}$ is the **Dirac measure** concentrated at $x_0$ and of total mass $\lambda$.

In $1d$, the Dirac delta distribution at $x_0 \in \mathbb{R}$ is usually schematically represented by an arrow based at $x_0$, of length one, and pointing upward. The height of the arrow is usually rescaled by a factor $|\lambda| \in \mathbb{R}$ to schematically represent the distribution $\lambda\delta_{x_0}$ with $\lambda \in \mathbb{C}$ (cf. **Figure 8.1**). In a measure-theoretic context (cf. **Example 8.22**), the scalar $\lambda$ represents the total mass concentrated at $x_0$. ⋯

**Example 8.6.** (E. T. WHITTAKER, C. SHANNON) The Dirac **comb**, also known as an **impulse train**, **sampling function**, or as **Shah** function because its «graph» resembles the shape of the Cyrillic letter sha «Ш», is very popular for its applications to sampling and aliasing. Indeed, it is at the heart of Whittaker–Shannon interpolation formula. Formally, the Dirac comb is an infinite series of Dirac delta distributions spaced at equal distance $T > 0$ (called the *period* of the Dirac comb). The definition of period of distribution and series of distributions will be given later on; for the purpose of this example it is not necessary to handle this right now.

Given a positive *period* $T > 0$, the Dirac comb $Ш_T$ is the linear form on $C_c^\infty(\mathbb{R}, \mathbb{C})$ defined by

$$Ш_T\colon \varphi \in C_c^\infty(\mathbb{R}, \mathbb{C}) \mapsto \sum_{n \in \mathbb{Z}} \varphi(nT) \in \mathbb{C}. \tag{8.5}$$

Clearly, for every compact subset $K \in \mathfrak{K}_{\mathbb{R}}$ we have (with $\mathbb{Z}T = \{z \in \mathbb{R} :: z = nT \text{ for some } n \in \mathbb{Z}\}$)

$$|\langle Ш_T, \varphi \rangle| \leqslant \sum_{n \in \mathbb{Z}} |\varphi(nT)| = \sum_{x \in K \cap \mathbb{Z}T} |\varphi(x)| \leqslant c_K \sup_{x \in K} |\varphi(x)| = c_K\, \mathfrak{p}_{K, m_K}(\varphi),$$

where $c_K := \#(K \cap \mathbb{Z}T)$ and $m_K = 0$. Note that $c_K$ really depends on $K \in \mathfrak{K}_{\mathbb{R}}$ while $m_K \equiv 0$ does not. Therefore, $Ш_T \in \mathcal{D}'(\mathbb{R})$ is a distribution of finite order (cf. **Definition 8.8**) and, more precisely, a distribution of order zero. ⋯

**Example 8.7.** (MULTIPOLES) Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $x_0 \in \Omega$. For a multi-index $\beta \in \mathbb{N}^N$ we consider the linear form $\delta_{x_0}^{\beta}\colon \varphi \in C_c^\infty(\Omega, \mathbb{C}) \mapsto \partial^{\beta}\varphi(x_0) \in \mathbb{C}$. It is simple to show that $\delta_{x_0}^{\beta}$ is a distribution on $\Omega$. Indeed, for any compact subset $K \in \mathfrak{K}_{\Omega}$ there exists a positive constant $c_K := 1$

and a nonnegative integer $m_K := |\beta| \in \mathbb{N}$ such that for all $\varphi \in C_K^\infty(\Omega, \mathbb{C})$

$$|\langle \delta_{x_0}^{\beta}, \varphi \rangle| \equiv |\partial^{\beta}\varphi(x_0)| \leqslant c_K\, \mathfrak{p}_{K,m_K}(\varphi) \equiv \sup_{|\alpha|\leqslant|\beta|} \sup_{x\in K} |\partial^{\alpha}\varphi(x)|. \tag{8.6}$$

Indeed, if $x_0 \notin K$ then $|\langle \delta_{x_0}^{\beta}, \varphi \rangle| = 0$, while if $x_0 \in K$ then $|\partial^{\beta}\varphi^{(n)}(x_0)| \leqslant \mathfrak{p}_{K,|\beta|}(\varphi)$. Note that both $c_K$ and $m_K$ do not depend on $K \in \mathfrak{K}_\Omega$. In particular, since $m_K$ does not depend on $K \in \mathfrak{K}_\Omega$, $\delta_{x_0}^{\beta}$ is a distribution of order less than or equal to $|\beta|$.

The distribution $\delta_{x_0}^{\beta} \in \mathcal{D}'(\Omega)$, $\beta \in \mathbb{N}_0^N$, is called the **$\beta$-order derivative of the delta** (**distribution**) at $x_0$. In electromagnetism, such a type of distribution models the action of a magnetic multipole located at $x_0$. ⋯

## 8.2 Distributions of finite order

**8.8. Definition.** Let $k \in \mathbb{N}$ **($k \neq \infty$)**, we call distribution on $\Omega$ **of order less than or equal to** $k$, any continuous linear functional on the space $\mathcal{D}^k(\Omega)$. The vector space of all distributions on $\Omega$ having order less or equal than $k$ is nothing but the **dual space** of $\mathcal{D}^k(\Omega)$ and it is, therefore, denoted by $(\mathcal{D}^k)'(\Omega)$. In other words, by definition

$$(\mathcal{D}^k)'(\Omega) := (\mathcal{D}^k(\Omega))'. \tag{8.7}$$

When $k = 0$, we say that $(\mathcal{D}^0)'(\Omega)$ is the space of distributions of order zero. Recall that, formally, the **symbols** $(\mathcal{D}^\infty)'(\Omega)$ **and** $\mathcal{D}'(\Omega)$ denote the same thing.

It is important to stress that while it is true that $\mathcal{D}(\Omega) = \bigcap_{k\in\mathbb{N}} \mathcal{D}^k(\Omega)$, it is **not true** that $\mathcal{D}'(\Omega) \equiv \bigcup_{k\in\mathbb{N}} (\mathcal{D}^k)'(\Omega)$. Indeed, $\bigcup_{k\in\mathbb{N}_0} (\mathcal{D}^k)'(\Omega)$ is nothing but the set of distributions having finite order and, in general, one has

$$\textstyle\bigcup_{k\in\mathbb{N}} (\mathcal{D}^k)'(\Omega) \subseteq \mathcal{D}'(\Omega). \tag{8.8}$$

The inclusion is **strict**, as the next example shows. Any element of $\mathcal{D}'(\Omega) \setminus \bigcup_{k\in\mathbb{N}} (\mathcal{D}^k)'(\Omega)$ is referred to as a **distribution of infinite order**.

**Example 8.9.** (Taylor-type distribution) Consider the linear form

$$Ш : \varphi \in C_c^\infty(\mathbb{R}) \mapsto \sum_{n\in\mathbb{N}} \partial^n \varphi(n) \in \mathbb{C}. \tag{8.9}$$

It is simple to show that Ш is a distribution on $\mathbb{R}$. Indeed, for any compact subset $K$ in $\mathbb{R}$, there exists a positive constant $c_K$ and a nonnegative integer $m_K \in \mathbb{N}$ such that for all $\varphi \in C_K^\infty(\Omega)$

$$|\langle Ш, \varphi \rangle| \leqslant \sum_{n\in\mathbb{N}\cap K} |\partial^n \varphi(n)| \leqslant c_K\, \mathfrak{p}_{K,m_K}(\varphi) \equiv c_K \sup_{\alpha\leqslant m_K} \sup_{x\in K} |\partial^{\alpha}\varphi(x)|. \tag{8.10}$$

Indeed, it is sufficient to set $c_K := \#(K \cap \mathbb{N})$ and $m_K = \max\{\alpha \in \mathbb{N} :: \alpha \in K \cap \mathbb{N}\}$. The previous inequality shows that $Ш \in \mathcal{D}'(\mathbb{R})$ is a distribution. However, $m_K$ depends on $K \in \mathfrak{K}_\mathbb{R}$, and we cannot conclude that Ш is a distribution of finite order. We cannot conclude, from the previous argument, that Ш has *infinite* order (because, maybe, an estimate different from the one we used tells us that Ш has finite order), although our intuition goes in that direction. In fact, as we now prove, Ш **has infinite order**. In this regard, we need the following observation.

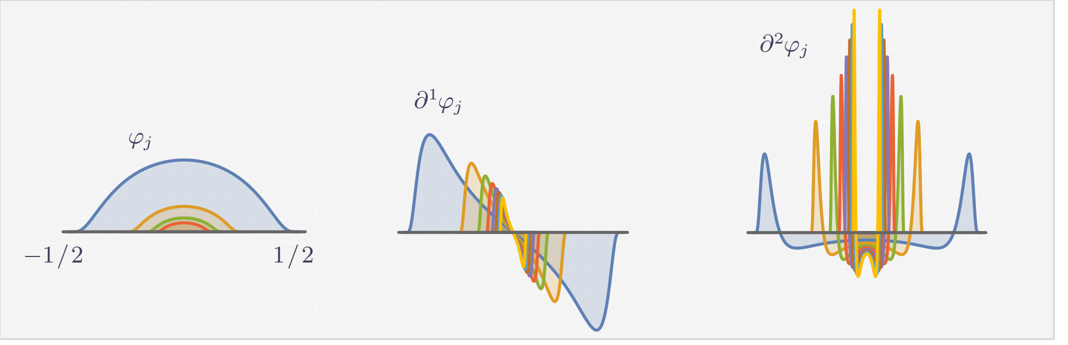



**Figure 8.3.** For every $j \in \mathbb{N}$ we define the function $\varphi_j : x \in \mathbb{R} \mapsto \varphi_j(x) := j^{-(k+1/2)}\varphi(jx)$ where $\varphi$ is in $\mathfrak{D}_K(\mathbb{R}) \subseteq \mathfrak{D}(\mathbb{R})$ with $K := [-1/2, 1/2]$. In the pitcure, we sketch the case $k = 1$. Note that $\sup_{x\in\mathbb{R}} |\partial^2\varphi_j(x)| \to +\infty$ when $j \to \infty$, i.e., the set $\{\varphi_j\}_{j\in\mathbb{N}}$ is unbounded in $\mathfrak{D}^2(\mathbb{R})$.

**Claim:** *For every $k \in \mathbb{N}$ there exists a sequence $(\varphi_j)_{j\in\mathbb{N}}$ in $\mathfrak{D}_K(\mathbb{R}) \subseteq \mathfrak{D}(\mathbb{R})$ with $K := [-1/2, 1/2]$, such that*

$$\varphi_j \to 0 \text{ in } \mathfrak{D}^k(\mathbb{R}) \quad \textbf{but} \quad |\partial^{k+1}\varphi_j(0)| \to +\infty \text{ in } \bar{\mathbb{R}}. \tag{8.11}$$

In particular, $(\varphi_j)_{j\in\mathbb{N}}$ converges in $\mathfrak{D}^k(\mathbb{R})$ but not in $\mathfrak{D}^{k+1}(\mathbb{R})$.

**8.10. Remark.** The construction in the proof allows generating sequences $(\varphi_j)_{j\in\mathbb{N}}$ satisfying the claim, from a quite arbitrary $\varphi$ in $\mathfrak{D}_K(\mathbb{R})$. Precisely, given any $\varphi \in \mathfrak{D}_K(\mathbb{R})$, if we know that $\partial^{k+1}\varphi(0) \neq 0$ then we can build a sequence $(\varphi_j)_{j\in\mathbb{N}}$ such that $\varphi_j$ converges in $\mathfrak{D}^k(\mathbb{R})$ but not in $\mathfrak{D}^{k+1}(\mathbb{R})$ because the modulus of its $k$-th order derivative in $0$ explodes to infinity.

**PROOF.** Let $\varphi$ be in $\mathfrak{D}_K(\mathbb{R})$. For every $j \in \mathbb{N}$ we define the function

$$\varphi_j : x \in \mathbb{R} \mapsto \varphi_j(x) := \frac{1}{j^{k+\frac{1}{2}}}\varphi(jx).$$

Since $\operatorname{supp}_{\mathbb{R}} \varphi \subseteq K = [-1/2, 1/2]$, for a given $j \in \mathbb{N}$ we have that $\varphi_j(x) = 0$ whenever $|x| > \frac{1}{2j}$. Therefore

$$\operatorname{supp}_{\mathbb{R}} \varphi_j \subseteq K_j := \left[-\frac{1}{2j}, \frac{1}{2j}\right] \quad \forall j \in \mathbb{N}.$$

Note that $K_1 \equiv K$ and $(K_j)_{j\in\mathbb{N}}$ is decreasing:

$$K_1 \supseteq K_2 \supseteq \ldots \supseteq K_j \supseteq \ldots.$$

Hence, $\operatorname{supp}_{\mathbb{R}} \varphi_j \subseteq K$ for every $j \in \mathbb{N}$. In fact, $(\varphi_j)_{j\in\mathbb{N}}$ is a sequence in $\mathfrak{D}_K(\mathbb{R})$. But this implies that, for every $h \in \mathbb{N}$, also the sequence $(\partial^h\varphi_j)_{j\in\mathbb{N}}$ of its $k$-th order derivatives is in $\mathfrak{D}_K(\mathbb{R})$. A direct computation shows that

$$\partial^h\varphi_j(x) = \frac{1}{j^{(k-h)+1/2}}\partial^h\varphi(jx). \tag{8.12}$$

The previous relation (8.12) shows that if $h \leqslant k$ the sequence $(\partial^h\varphi_j)_{j\in\mathbb{N}}$ uniformly converges to $0$ in $\mathbb{R}$. Indeed, if $h \leqslant k$ then $j^{-((k-h)+1/2)} \to 0$ when $j \to \infty$ and, therefore,

$$\sup_{x\in\mathbb{R}} |\partial^h\varphi_j(x)| \leqslant \frac{1}{j^{(k-h)+1/2}} \sup_{x\in\mathbb{R}} |\partial^h\varphi(x)| \xrightarrow{j\to\infty} 0.$$

In particular, $\varphi_j \to 0$ in $\mathfrak{D}^k(\mathbb{R})$.

On the other hand, if $h = k+1$ then

$$|\partial^{k+1}\varphi_j(0)| = \sqrt{j}\,|\partial^{k+1}\varphi(0)|$$

and, therefore, if we choose $\varphi$ such that $\partial^{k+1}\varphi(0) \neq 0$ then $|\partial^{k+1}\varphi_j(0)| \to +\infty$ when $j \to \infty$. Clearly, this implies that $\sup_{x\in\mathbb{R}} |\partial^{k+1}\varphi_j(x)| \to +\infty$ when $j \to \infty$, because of

$$\sup_{x\in\mathbb{R}} |\partial^{k+1}\varphi_j(x)| = \sqrt{j}\sup_{x\in\mathbb{R}} |\partial^{k+1}\varphi(jx)| \geqslant \sqrt{j}\,|\partial^{k+1}\varphi(0)|.$$

In particular, the set $\{\varphi_j\}_{j\in\mathbb{N}}$ is unbounded in $\mathcal{D}^{k+1}(\mathbb{R})$ due to **Proposition 6.58**.

Note that, in the previous claim, the assumption that $K$ is centered around the origin plays no special role. Indeed, the following result holds.

**8.11. Proposition.** *Let $k \in \mathbb{N}$. For every $a \in \mathbb{R}$ there exists a sequence $(\varphi_j)_{j\in\mathbb{N}}$ in $\mathcal{D}_{a+K}(\mathbb{R}) \subseteq \mathcal{D}(\mathbb{R})$ with $K := [-1/2, 1/2]$ and $a + K := [a - 1/2, a + 1/2]$, such that*

$$\varphi_j \to 0 \text{ in } \mathcal{D}^k(\mathbb{R}) \quad \textbf{but} \quad \partial^{k+1}\varphi_j(a) \to +\infty \text{ in } \bar{\mathbb{R}}. \tag{8.13}$$

After that, suppose that Ш has finite order $k \in \mathbb{N}$. We consider the compact interval of $\mathbb{R}$ defined by $K_k := (k+1) + K = [(k+1) - 1/2, (k+1) + 1/2]$. Note that

$$\mathbb{N} \cap K_k = \{k+1\}. \tag{8.14}$$

As Ш has order less than or equal to $k$, there exists a constant $c_k > 0$ such that

$$|\langle Ш, \varphi\rangle| \leqslant c_k \sup_{\alpha\leqslant k}\sup_{x\in K_k} |\partial^\alpha\varphi(x)| \tag{8.15}$$

for any $\varphi \in \mathcal{D}^k_{K_k}(\mathbb{R})$. On the other hand, taking into account (8.14), for every $\varphi \in \mathcal{D}^k_{K_k}(\mathbb{R})$ we have

$$\langle Ш, \varphi\rangle = \sum_{n\in\mathbb{N}} \partial^n\varphi(n) = \sum_{n\in\mathbb{N}\cap K_k} \partial^n\varphi(n) \overset{(8.14)}{=} \partial^{k+1}\varphi(k+1).$$

Hence, combining the previous relation with (8.15), we get that if Ш is a distribution of order less than or equal to $k$, there holds

$$|\langle Ш, \varphi\rangle| = |\partial^{k+1}\varphi(k+1)| \leqslant c_{K_k} \sup_{\alpha\leqslant k}\sup_{x\in K_k} |\partial^\alpha\varphi(x)|$$

for every $\varphi \in \mathcal{D}^k_{K_k}(\mathbb{R})$. In particular,

$$|\langle Ш, \varphi\rangle| = |\partial^{k+1}\varphi_j(k+1)| \to 0$$

for every sequence $(\varphi_j)_{j\in\mathbb{N}} \to 0$ in $\mathcal{D}^k_{K_k}(\mathbb{R})$. But this cannot be possible. Just consider the sequence $(\varphi_j)_{j\in\mathbb{N}}$ of **Proposition 8.11** with $a := k+1$.

### 8.2.1. Injection of $(\mathcal{D}^k)'(\Omega)$ into $\mathcal{D}'(\Omega)$

**8.12. Proposition.** *Let $k, h \in \bar{\mathbb{N}}$, and $k \leqslant h$. There exists a continuous canonical injection of $(\mathcal{D}^k)'(\Omega)$ into $(\mathcal{D}^h)'(\Omega)$. In particular, any distribution of order less or equal than $k$ can be identified with a distribution.*

**8.13. Remark.** The existence of a continuous injection $j^\top$ of $(\mathcal{D}^k)'(\Omega)$ into $(\mathcal{D}^h)'(\Omega)$ means that $j^\top(S) = j^\top(T)$ if and only if $S = T$ with $j^\top(S), j^\top(T) \in (\mathcal{D}^h)'(\Omega)$. Therefore $(\mathcal{D}^k)'(\Omega)$ can be identified with the subspace $j^\top((\mathcal{D}^k)'(\Omega)) \subseteq (\mathcal{D}^h)'(\Omega)$.

**Proof.** For integers $0 \leqslant k \leqslant h \leqslant \infty$ one has $C_c^h(\Omega) \trianglelefteq C_c^k(\Omega)$. Moreover (cf. **Proposition 6.60**), the topology of $\mathfrak{D}^h(\Omega) = (C_c^h(\Omega), \tau_{\mathrm{LF}}^h)$ is finer than the subspace topology induced on $C_c^h(\Omega)$ by $\mathfrak{D}^k(\Omega)$. Therefore, the canonical injection $j\colon \mathfrak{D}^h(\Omega) \hookrightarrow \mathfrak{D}^k(\Omega)$ is continuous. But then, **Theorem 6.61** shows that $C_c^h(\Omega) \equiv j(C_c^h(\Omega))$ is dense in $\mathfrak{D}^k(\Omega)$ and the proof follows from **Theorem 7.38** concerning the injection of dual spaces. More precisely: the transpose map $j^\top$ of $j$ injects $(\mathfrak{D}^k)'(\Omega)$ into $(\mathfrak{D}^h)'(\Omega)$; moreover every linear form $T_h$ on $C_c^h(\Omega)$, which is continuous on $C_c^h(\Omega)$ for the topology of $\mathfrak{D}^k(\Omega)$, is extendable to a continuous linear form $T_h^\#$ on $\mathfrak{D}^k(\Omega)$ and $j^\top(T_h^\#) = T_h$. ■

We have the following useful characterization of distributions of finite order.

**8.14. Proposition.** *Let $T$ be a* **linear form** *on $C_c^\infty(\Omega)$. The following three assertions are equivalent:*

***i.*** *The linear form $T$ is identifiable to a distribution of order less than or equal to $k$.*

***ii.*** *The linear form $T$ is continuous on $(C_c^\infty(\Omega), \tau_{\mathrm{LF}}^k)$, i.e., on $C_c^\infty(\Omega)$ endowed with the subspace topology induced by $\mathfrak{D}^k(\Omega)$.*

***iii.*** *For every compact subset $K$ of $\Omega$, there exists a constant $c_K$ (depending only on the given compact set $K$) such that*

$$\mathfrak{p}_\varphi(T) := |\langle T, \varphi\rangle| \leqslant c_K\, \mathfrak{p}_{K,k}(\varphi) \quad \forall \varphi \in \mathfrak{D}_K(\Omega).$$

*Recall that $\mathfrak{p}_{K,k}(\varphi) := \sup_{|\boldsymbol{\alpha}| \leqslant k} \sup_{x\in K} |\partial^{\boldsymbol{\alpha}} \varphi(x)|$.*

**Proof.** It is clear that ***i.*** implies ***ii.***, because point ***i.*** means that there exists an extension $\tilde{T}$ of $T$ to $\mathfrak{D}^k(\Omega)$ and such an extension is continuous in $\mathfrak{D}^k(\Omega)$, i.e., $\tilde{T} \in (\mathfrak{D}^k)'(\Omega)$.

That ***ii.*** implies ***i.*** follows from **Theorem 7.38** concerning the injection of dual spaces. More precisely, the linear form $T$ is, for the time being, defined just on $C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $\mathfrak{D}^k(\Omega)$ and $T$, by hypothesis, is continuous on $(C_c^\infty(\Omega), \tau_{\mathrm{LF}}^k)$, there exists a unique extension of $T$ to $\mathfrak{D}^k(\Omega)$ by the **principle of extension by continuity** (cf. **Theorem 3.36**).

Finally, the equivalence of ***ii.*** and ***iii.*** is a consequence of **Proposition 6.59** as soon as we note that if $T$ is continuous on $(C_c^\infty(\Omega), \tau_{\mathrm{LF}}^k)$ then (the unique extension of) $T$ is continuous on $\mathfrak{D}^k(\Omega)$ and therefore, from **Proposition 6.59**, we have that $T$ is continuous on $\mathfrak{D}_K^k(\Omega)$ for every compact set $K \subseteq \Omega$. ■

## 8.3 | Radon Measures

In **Definition 6.32** we defined Radon measures as the dual space of $\mathfrak{K}(\Omega)$ with $\Omega$ a $\sigma$-locally compact Hausdorff space. Since every open subset $\Omega \subseteq \mathbb{R}^N$ is a $\sigma$-locally compact Hausdorff space, we can identify (due to **Proposition 8.14**) the space $\mathfrak{M}(\Omega) := \mathfrak{K}'(\Omega)$ with the set of distributions of order less than or equal to zero on $\Omega$. In fact, in the theory of distributions, one usually gives the following definition.

**8.15. Definition.** We call Radon measure on $\Omega$ every distribution of order zero on $\Omega$, i.e., any element of $(\mathfrak{D}^0)'(\Omega)$.

According to the characterization given in **Proposition 8.14**, a linear form $\mu$ on $C_c^\infty(\Omega)$ is (identifiable with) a Radon measure, if for every compact $K \in \mathfrak{K}_\Omega$ there exists a constant $c_K$, depending on $K$, such that $|\langle \mu, \varphi\rangle| \leqslant c_K \sup_{x\in K} |\varphi(x)|$ for every $\varphi \in \mathfrak{K}_K(\Omega)$. The set of Radon measures on $\Omega$ is, essentially, nothing but the dual of the locally convex space $\mathfrak{K}(\Omega) \equiv \mathfrak{D}^0(\Omega)$ and will be denoted by $(\mathcal{D}^0)'(\Omega)$ or by $\mathcal{K}'(\Omega)$ or by $\mathcal{M}(\Omega)$. For $\mu \in \mathcal{K}'(\Omega)$ and $\varphi \in \mathcal{K}(\Omega)$, the value of $\mu$ in $\varphi$ will be denoted by one of the following symbols

$$\mu(\varphi), \quad \langle \mu, \varphi \rangle, \quad \int_\Omega \varphi \, \mathrm{d}\mu, \quad \int_\Omega \varphi(x) \, \mathrm{d}\mu(x). \tag{8.16}$$

For the sake of concreteness, we specialize **Proposition 8.14** to the case of distributions of order zero.

**8.16. Proposition.** *Let $\mu$ be a* **linear form** *on $C_c^\infty(\Omega)$. The following three assertions are equivalent:*

*i.* *The linear form $\mu$ is (extendable to) a Radon measure on $\Omega$.*

*ii.* *The linear form $\mu$ is continuous on $(C_c^\infty(\Omega), \tau_{\mathrm{LF}})$, i.e., when $C_c^\infty(\Omega)$ is endowed with the subspace topology induced by $\mathcal{K}(\Omega)$.*

*iii.* *For every compact subset $K$ of $\Omega$, there exists a constant $c_K$ (depending only on the given compact set $K$) such that*

$$\mathfrak{p}_\varphi(T) := |\langle T, \varphi \rangle| \leqslant c_K \, \mathfrak{p}_{K,0}(\varphi) \quad \forall \varphi \in \mathcal{D}_K(\Omega).$$

*Recall that $\mathfrak{p}_{K,0}(\varphi) := \sup_{x \in K} |\varphi(x)|$.*

**8.17. Definition.** The weak star dual topology on $\mathcal{K}'(\Omega)$ is called the **topology of *vague* convergence**. A generalized sequence $\{\mu_\lambda\}_{\lambda \in \Lambda}$ vaguely converges to a measure $\mu$ if, and only if, for every $\varphi \in \mathcal{K}(\Omega)$ the generalized *numerical* sequence $\langle \mu_\lambda, \varphi \rangle$ converges to $\langle \mu, \varphi \rangle$.

**8.18. Remark.** The vague topology on $\mathcal{K}'(\Omega)$ is finer than the one induced by the weak dual topology of $\mathcal{D}'(\Omega)$. Indeed the injection of $\mathcal{K}'(\Omega)$ into $\mathcal{D}'(\Omega)$ is continuous. ⋯

### 8.3.1. Positive Radon Measures

We now show that any positive linear form on $C_c(\Omega, \mathbb{C})$ is a positive Radon measure. We start with some definitions.

**8.19. Definition.** Let $\mu$ be a radon measure on $\Omega$. We say that $\mu$ is a **real Radon measure** if for every $\varphi \in \mathcal{K}(\Omega, \mathbb{C})$ such that $\Im(\varphi) \equiv 0$ we have $\Im(\langle \mu, \varphi \rangle) = 0$, i.e., if for every *real-valued* function $\varphi \in \mathcal{K}(\Omega, \mathbb{C})$, the number $\langle \mu, \varphi \rangle$ is real. We say that $\mu$ is **positive** if, for every $\varphi \in \mathcal{K}(\Omega, \mathbb{C})$ which is positive (possibly null at some or all point in $\Omega$) the value $\langle \mu, \varphi \rangle$ is positive (or null). In symbols, $\mu$ is positive if, and only, if $\langle \mu, \varphi \rangle \geqslant 0$ for every $\varphi \in C_c(\Omega, \mathbb{R})$ such that $\varphi \geqslant 0$ in $\Omega$. We say that $\mu$ is **monotone** if for every real-valued $\varphi, \psi \in \mathcal{K}(\Omega, \mathbb{C})$, there holds that $\langle \mu, \varphi \rangle \leqslant \langle \mu, \psi \rangle$ whenever $\varphi \leqslant \psi$ in $\Omega$.

The term «positive» in the context of Radon measures is a synonym of «non-negative», that is of $\geqslant 0$ rather than $> 0$.

The notion of positive Radon measure is a particular case of the more general notion of positive linear form on an ordered vector space. For example, one says that a linear form $L$ on a (partially) ordered vector space $(V, \leqslant)$ is positive if $L(v) \geqslant 0$ whenever $v \geqslant 0$. Since $C_c(\Omega, \mathbb{C})$ is (partially) ordered by the relation $\varphi \leqslant \psi$ if, and only if, $\varphi, \psi$ are real-valued and $\varphi(x) \leqslant \psi(x)$ for every $x \in \Omega$, the concept of positive Radon measure is an instance of the concept of positive linear form. Also the concept of *real* Radon measure extends to (not necessarily continuous) linear form on $\mathcal{K}(\Omega, \mathbb{C})$. For example, we say that $\mu$ is a **real** linear form on $C_c(\Omega, \mathbb{C})$ if for every $\varphi \in C_c(\Omega, \mathbb{C})$ such that $\Im(\varphi) \equiv 0$ we have $\Im(\langle \mu, \varphi \rangle) = 0$. We say that $\mu$ is a **monotone** linear form on $C_c(\Omega, \mathbb{C})$ if for every $\varphi, \psi \in C_c(\Omega, \mathbb{C})$ we have $\langle \mu, \varphi \rangle \leqslant \langle \mu, \psi \rangle$ as soon as $\varphi \leqslant \psi$ in $\Omega$.

For any real linear form on $C_c(\Omega,\mathbb{C})$, the following properties hold.

**8.20. Proposition.** *Let $\mu$ be a real linear form on $C_c(\Omega,\mathbb{C})$. Then $\mu$ is monotone, if and only if, $\mu$ is positive. Moreover, if is $\mu$ is monotone (or positive) then*

$$\forall\varphi\in C_c(\Omega,\mathbb{R})\quad |\langle\mu,\varphi\rangle|\leqslant\langle\mu,|\varphi|\rangle. \tag{8.17}$$

*It follows, that*

$$\forall\varphi,\psi\in C_c(\Omega,\mathbb{C})\quad |\varphi|\leqslant\psi\quad\Rightarrow\quad|\langle\mu,\varphi\rangle|\leqslant c_{\mathbb{K}}\langle\mu,\psi\rangle. \tag{8.18}$$

*with $c_{\mathbb{K}}=1$ if $\varphi$ is real-valued and $c_{\mathbb{K}}=2$ if $\varphi$ is complex-valued. Note that, instead, $\psi$ is necessarily real-valued.*

**PROOF.** Let $\mu\in[C_c(\Omega,\mathbb{C})]^*$ be a real linear form. Assume that $\mu$ is monotone. Since $0\in C_c(\Omega,\mathbb{R})$, we have that if $\psi\geqslant 0$, then

$$\langle\mu,\psi\rangle\geqslant\langle\mu,0\rangle=0.$$

Conversely, let $\mu$ be a positive linear form and assume that $\varphi\leqslant\psi$. Then $\psi-\varphi\geqslant 0$ and, therefore, $0\leqslant\langle\mu,\psi-\varphi\rangle=\langle\mu,\psi\rangle-\langle\mu,\varphi\rangle$. Hence $\langle\mu,\psi\rangle\geqslant\langle\mu,\varphi\rangle$.

Next, we prove (8.17). For that, we observe that since $\varphi$ is real-valued, we have

$$-|\varphi|\leqslant\varphi\leqslant|\varphi|\quad\text{in }\Omega.$$

By monotonicity and linearity we infer that $-\langle\mu,|\varphi|\rangle\leqslant\langle\mu,\varphi\rangle\leqslant\langle\mu,|\varphi|\rangle$, that is $|\langle\mu,\varphi\rangle|\leqslant\langle\mu,|\varphi|\rangle$. This proves (8.17). Next, assume that $\varphi,\psi\in C_c(\Omega,\mathbb{C})$ and $|\varphi|\leqslant\psi$. We write $\varphi:=\Re e\varphi+i\Im\varphi$ and note that, due to (8.17), we have

$$\begin{aligned}|\langle\mu,\varphi\rangle| &\leqslant |\langle\mu,\Re e\varphi\rangle|+|\langle\mu,\Im\varphi\rangle|\\ &\overset{(8.17)}{\leqslant} \langle\mu,|\Re e\varphi|\rangle+\langle\mu,|\Im\varphi|\rangle\\ &\leqslant c_{\mathbb{K}}\langle\mu,\psi\rangle.\end{aligned}$$

The last equality following from the monotonicity of $\mu$ and the assumed relation $|\varphi|\leqslant\psi$. Indeed, trivially, $|\Re e\varphi|\leqslant|\varphi|\leqslant\psi$ and $|\Im\varphi|\leqslant|\varphi|\leqslant\psi$ in $\Omega$. ■■■■

**8.21. Proposition.** *Let $\mu$ be a* **linear form** *on $C_c(\Omega,\mathbb{C})$, i.e., an element of the* **algebraic** *dual of $C_c(\Omega,\mathbb{C})$. If $\mu$ is positive on $C_c(\Omega,\mathbb{C})$ then $\mu$ is continuous on $\mathcal{K}(\Omega,\mathbb{C})$. In other words, the set of* **positive linear forms** *on $C_c(\Omega,\mathbb{C})$ coincides with the set $\mathcal{K}'(\Omega,\mathbb{C})$ of* **positive Radon measures** *on $\Omega$. In particular, every positive linear form on $C_c(\Omega,\mathbb{C})$ is a distribution of order zero.*

**PROOF.** Let $K\in\mathfrak{K}_\Omega$ be a compact subset of $\Omega$. Let $\varphi\in\mathcal{K}_K(\Omega,\mathbb{C})$. Clearly,

$$|\varphi(x)|\leqslant\mathfrak{p}_{K,0}(\varphi)=\sup_{x\in K}|\varphi(x)|\quad\forall x\in\Omega.$$

Consider an Urysohn cut-off function $\psi$, i.e., a nonnegative function which is equal to $1$ on $K$ and that belongs to $\mathcal{K}(\Omega,\mathbb{R})$. The existence of such a function is guaranteed by **Urysohn's separation Lemma** ?. We then have

$$|\varphi(x)|\leqslant\psi(x)\mathfrak{p}_{K,0}(\varphi)\quad\forall x\in\Omega.$$

Indeed, the previous relations reduce to $|\varphi(x)|\leqslant\sup_K|\varphi|$ in $K$ and to $\sup_K|\varphi|\psi(x)\geqslant 0$ in $\Omega\setminus K$ because of $\operatorname{supp}_\Omega\varphi\subseteq K$. But then, from (8.18), we infer that

$$|\langle\mu,\varphi\rangle|\leqslant(c_{\mathbb{K}}\langle\mu,\psi\rangle)\,\mathfrak{p}_{K,0}(\varphi).$$

Since $\langle \mu, \psi \rangle$ depends only on $K$ and $\mathfrak{p}_{K,0}$ is the norm defining the topology of $\mathcal{K}_K(\Omega, \mathbb{C})$, the linear form $\mu$ is continuous on $\mathcal{K}_K(\Omega, \mathbb{C})$. The arbitrariness of $K \in \mathfrak{K}_\Omega$ concludes the proof of the continuity of $\mu$ on $\mathcal{K}(\Omega, \mathbb{C})$.

**Example 8.22.** (Dirac measure) Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $x_0 \in \Omega$. The Dirac delta concentrated at $x_0$, defined by (cf. **Example 8.5**)

$$\delta_{x_0} \colon \varphi \in C_c^\infty(\Omega, \mathbb{C}) \mapsto \varphi(x_0) \in \mathbb{C},$$

can be trivially extended to a linear form on $C_c(\Omega, \mathbb{C})$. As an element of the algebraic dual of $C_c(\Omega, \mathbb{C})$, $\delta_{x_0}$ is positive. Therefore $\delta_{x_0}$ is a positive Radon measure on $\Omega$.

**Example 8.23.** (Dirac comb) Given a *period* $T > 0$, the Dirac comb distribution $\text{Ш}_T$, defined by (cf. **Example 8.6**)

$$\text{Ш}_T \colon \varphi \in C_c^\infty(\mathbb{R}, \mathbb{C}) \mapsto \sum_{n \in \mathbb{Z}} \varphi(nT) \in \mathbb{C}, \tag{8.19}$$

can be trivially extended to a linear form on $C_c(\mathbb{R}, \mathbb{C})$. As an element of the algebraic dual of $C_c(\mathbb{R}, \mathbb{C})$, $\text{Ш}_T$ is positive. Therefore $\text{Ш}_T$ is a positive Radon measure on $\mathbb{R}$.

### 8.3.2. Regular distributions

In this section, we show that if $\Omega$ is an open subset of $\mathbb{R}^N$, then the elements of $L^1_{\text{loc}}(\Omega)$ can be identified to a subspace of the space of distributions of order zero (a Radon measure): $L^1_{\text{loc}}(\Omega) \trianglelefteq \mathcal{K}'(\Omega)$. When an $L^1_{\text{loc}}(\Omega)$ element is identified to a Radon measure, it is referred to as a regular distribution. In other words, the space of regular distributions is nothing but the subspace of $\mathcal{K}'(\Omega)$ consisting of those Radon measures that admit a $L^1_{\text{loc}}(\Omega)$ representative (via a canonical injection), in the precise sense specified by the following result.

**8.24. Theorem.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $f \in L^1_{\text{loc}}(\Omega)$. The following assertions hold:*

***i.*** *The* **linear form** *$T_f$ on $C_c(\Omega)$ defined by*

$$T_f \colon \varphi \mapsto \int_\Omega f(x)\varphi(x)\,\mathrm{d}x,$$

*is a Radon measure on $\Omega$ (a distribution of order zero).*

***ii.*** *If $T_f$ is the null measure, then necessarily $f \equiv 0$ a.e. in $\Omega$.*

*Conditions **i.** and **ii.** ensure that the map $T$ defined by*

$$T \colon f \in L^1_{\text{loc}}(\Omega) \mapsto T_f \in \mathcal{K}'(\Omega)$$

*is a (linear) injection of $L^1_{\text{loc}}(\Omega)$ into $\mathcal{K}'(\Omega)$. It is called the* **canonical injection** *of $L^1_{\text{loc}}(\Omega)$ into $\mathcal{K}'(\Omega)$.*

***iii.*** *The canonical injection of $L^1_{\text{loc}}(\Omega)$ into $\mathcal{K}'(\Omega)$ is continuous.*

**8.25. Remark.** Thanks to the previous theorem, we can identify any element $f \in L^1_{\text{loc}}(\Omega)$ with the distribution $T_f \in \boldsymbol{\mathcal{K}}'(\Omega)$. Indeed, if $f_1, f_2 \in L^1_{\mathrm{loc}}(\Omega)$ and $T_{f_1} = T_{f_2}$, then $T_{(f_1 - f_2)} \equiv 0$ and, therefore, $f_1 = f_2$ a.e. in $\Omega$. Thus, $f_1$ and $f_2$ represent the same element in $L^1_{\mathrm{loc}}(\Omega)$. We say that $T_f$ is the distribution (Radon measure) represented by $f$. Often, one uses the notation $[f]$ to denote the regular distribution associated with $f \in L^1_{\mathrm{loc}}(\Omega)$, i.e., one writes

$$\langle [f], \varphi \rangle = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x.$$

Some times one uses the notation $\langle f, \varphi \rangle$ as a replacement of the more correct ones $\langle T_f, \varphi \rangle$ and $\langle [f], \varphi \rangle$.

**8.26. Definition.** Any Radon measure (distribution of order zero) which admits a representative in $L^1_{\mathrm{loc}}(\Omega)$ is called a **regular distribution**. In other words, a distribution $T \in \boldsymbol{\mathcal{D}}'(\Omega)$ is called a regular distribution if there exists an element $f \in L^1_{\mathrm{loc}}(\Omega)$ such that $T \equiv T_f$.

For the proof of assertion ***ii.*** in **Theorem 8.24**, we need the **smooth** version of Urysohn's separation Lemma (cf. **Lemma ?**) that here we recall.

**8.27. Lemma. (Urysohn, Smooth separation Lemma)** *Let $(K, F)$ a* **compact-closed** *pair of $\mathbb{R}^N$. If the pair is disjoint, i.e., if $K \cap F = \emptyset$, then there exists a function $\psi \in C_c^\infty(\mathbb{R}^N)$ having the following three properties:*

***i.*** $0 \leqslant \psi(x) \leqslant 1$ *for every $x \in \mathbb{R}^N$;*

***ii.*** $\psi \equiv 0$ *on a neighborhood of $F$;*

***iii.*** $\psi \equiv 1$ *on a neighborhood of $K$.*

*In particular, given an open set $\Omega$ of $\mathbb{R}^N$ and a compact subset $K$ of $\Omega$, there exists a* **Urysohn cut-off function** *$\psi \in C_c^\infty(\Omega)$, such that $0 \leqslant \psi(x) \leqslant 1$ for every $x \in \Omega$ and $\psi = 1$ on a* **compact neighborhood** *of $K$.*

○

**PROOF. (of Theorem 8.24)** ***i.*** Let $K \in \mathfrak{K}_\Omega$ be a compact subset of $\Omega$. For any $\varphi \in \boldsymbol{\mathcal{K}}_K(\Omega)$ we have

$$|T_f(\varphi)| \leqslant \mathfrak{p}_{K,0}(\varphi) \int_K |f(x)|\,\mathrm{d}x.$$

Therefore, for every $K \in \mathfrak{K}_\Omega$ there exists a nonnegative constant $c_K := \|f\|_{L^1(K)}$ such that

$$|T_f(\varphi)| \leqslant c_K \cdot \mathfrak{p}_{K,0}(\varphi) \quad \forall \varphi \in \boldsymbol{\mathcal{K}}_K(\Omega).$$

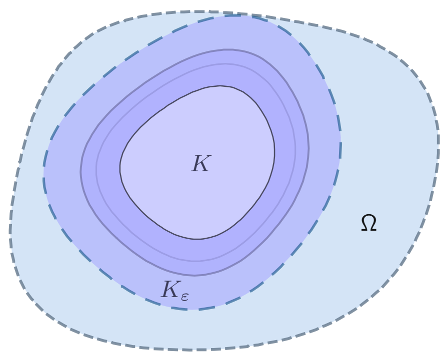



**Figure 8.4.** Let $K \in \mathfrak{K}_\Omega$ be a compact subset of $\Omega$. Let $\varepsilon < \operatorname{dist}(K, \Omega^{\complement})$. We construct a sequence $(\psi_j)_{j\in\mathbb{N}}$ of $C_c^\infty(\Omega)$ functions such that for every $j \in \mathbb{N}$, $0 \leqslant \psi_j(x) \leqslant 1$ in $\Omega$, $\psi_j \equiv 1$ in $K$, and $\psi_j \equiv 0$ in $\Omega \backslash K_{\varepsilon/j}$, where $K_{\varepsilon/j}$ is the compact neighborhood of $K$ of size $\varepsilon/j$. Namely, $K_{\varepsilon/j} := \{x \in \Omega :: \operatorname{dist}(x, K) < \varepsilon/j\}$.

The arbitrariness of the compact set $K$ proves the continuity of $T_f$ on $\mathcal{K}(\Omega)$.

***ii.*** Let $f \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\int_\Omega f(x)\varphi(x)\,\mathrm{d}x = 0$ for all $\varphi \in C_c(\Omega)$. We want to show that necessarily $f \equiv 0$ a.e. in $\Omega$. We prove a stronger result, the so-called *fundamental theorem of Calculus of Variations*. It states that if $f \in L^1_{\mathrm{loc}}(\Omega)$ is such that $\int_\Omega f(x)\varphi(x)\,\mathrm{d}x = 0$ for all $\varphi \in C_c^\infty(\Omega)$ then necessarily $f \equiv 0$ a.e. in $\Omega$[8.1]. We give two possible arguments.

**Proof 1 (via Urysohn's separation Lemma and Lebesgue differentiation theorem).** Let $K \in \mathfrak{K}_\Omega$ be a compact subset of $\Omega$. Let $\varepsilon < \operatorname{dist}(K, \Omega^{\complement})$. We consider the $G_\delta$ approximation of $K$ given by $(K_{\varepsilon/j})_{j\in\mathbb{N}}$ where $K_{\varepsilon/j}$ is the compact neighborhood of $K$ of size $\varepsilon/j$. Namely (cf. **Figure 8.4**):

$$K_{\varepsilon/j} := \{x \in \Omega :: \operatorname{dist}(x, K) < \varepsilon/j\}.$$

On such a decreasing approximation $(K_{\varepsilon/j})_{j\in\mathbb{N}}$ of $K$, we construct the sequence $(\psi_j)_{j\in\mathbb{N}}$ of $C_c^\infty(\Omega)$ functions such that for every $j \in \mathbb{N}$

$$0 \leqslant \psi_j(x) \leqslant 1 \quad \text{in } \Omega, \quad \psi_j \equiv 1 \text{ in } K, \quad \psi_j \equiv 0 \text{ in } \Omega \backslash K_{\varepsilon/j}.$$

Note that, since $(K_{\epsilon/j})_{j\in\mathbb{N}}$ is a decreasing sequence, we have

$$1_K \geqslant 1_{K_{\varepsilon/j}} \geqslant \psi_j \geqslant 1_K \quad \forall j \in \mathbb{N}. \tag{8.20}$$

The existence of such a family of functions is a consequence of the smooth Urysohn's separation **Lemma 8.27**. Note the following facts:

- When $j \to +\infty$, the sequence $1_{K_{\varepsilon/j}} \to 1_K$ pointwise in $\Omega$. Hence, by the sandwich lemma and (8.20), we have $(\psi_j)_{j\in\mathbb{N}} \to 1_K$ pointwise in $\Omega$. Thus, $(f\psi_j)_{j\in\mathbb{N}} \to f 1_K$ pointwise a.e. in $\Omega$.

- By the uniform bound in (8.20), the sequence $(f\psi_j)_{j\in\mathbb{N}}$ is dominated by the function $|f| 1_K$, which is integrable because, by assumption, $f \in L^1_{\mathrm{loc}}(\Omega)$.

By Lebesgue dominated convergence theorem, we infer that

$$\lim_{j\to\infty} \int_\Omega (f\psi_j)(x)\,\mathrm{d}x = \int_\Omega (f 1_K)(x)\,\mathrm{d}x. \tag{8.21}$$

By hypothesis $\int_\Omega f(x)\,\psi_j(x)\,\mathrm{d}x = 0$ for every $j \in \mathbb{N}$ and, therefore, by (8.21), $\|f\|_{L^1(K)} = 0$. The arbitrariness of $K$ ensures that $\|f\|_{L^1(K)} = 0$ for every $K \in \mathfrak{K}_\Omega$. By Lebesgue differentiation theorem,

8.1. The result we are going to prove is stronger because we are going to infer that $f \equiv 0$ by testing the distribution $T_f$ against elements in $C_c^\infty(\Omega)$ rather than in the superset $C_c(\Omega)$.

we conclude that $f\equiv 0$ a.e. in $\Omega$.

**Proof 2 (via regularization by convolution and Lebesgue differentiation theorem).** Let $K\in\mathfrak{K}_\Omega$ be a compact subset of $\Omega$. Let $\varepsilon<\operatorname{dist}(K,\Omega^\complement)$. We denote by $g:=f\cdot 1_{K_\varepsilon}$ the extension of $f$ to all of $\mathbb{R}^N$ which is equal to zero outside the compact neighborhood $K_\varepsilon$ of $K$. We denote by $(\theta_j)_{j\in\mathbb{N}}$ a regularizing sequence such that $\operatorname{supp}\theta_j\subset B(\varepsilon/j)$. Also, we denote by $g_j:=g*\theta_j$ the regularized of $g$ **at resolution** $\varepsilon/j$. By definition, one has

$$g_j(x)=\int_{\mathbb{R}^N}g(y)\,\theta_j(y-x)\,\mathrm{d}y,\quad (\text{for every } x\in\mathbb{R}^N).$$

We observe that for every $x\in K$ the support of $\tau_x\check{\theta}_j=\theta_j((\cdot)-x)$ is contained in $K+B(\varepsilon)$ and, therefore, in the compact neighborhood $K_\varepsilon$ where one has $g\equiv f$. It follows that

$$g_j(x)=\int_\Omega f(y)\,\theta_j(y-x)\,\mathrm{d}y,\quad (\text{for every } x\in K).$$

Since $\tau_x\check{\theta}_j\in C_c^\infty(\Omega)$ when $x\in K$ (regardless of $j\in\mathbb{N}$) and by hypothesis $\langle f,\varphi\rangle=0$ for every $\varphi\in C_c^\infty(\Omega)$, we have

$$g_j(x)=\langle f,\tau_x\check{\theta}_j\rangle=0\quad(\text{for every } x\in K).\tag{8.22}$$

On the other hand, by the **regularization theorem**, the regularized sequence $(g_j)_{j\in\mathbb{N}}\to g$ strongly in $L^1(\mathbb{R}^N)$. Whence $\|g-g_j\|_{L^1(K)}\leqslant\|g-g_j\|_{L^1(\mathbb{R}^N)}\to 0$. But on $K$ we have that $\|g-g_j\|_{L^1(K)}=\|g\|_{L^1(K)}$ because of (8.22) and, therefore, $\|g\|_{L^1(K)}=0$ for every compact subset $K$ of $\Omega$. By Lebesgue differentiation theorem, we conclude that $f\equiv 0$ a.e. in $\Omega$.

***iii.*** The injection of $L^1_{\mathrm{loc}}(\Omega)$ into $\boldsymbol{\mathcal{K}}'(\Omega)$ is manifestly continuous when the space $\boldsymbol{\mathcal{K}}'(\Omega)$ is endowed with the **vague topology** (cf. **Definition 8.17**). Indeed, for any $\varphi\in\boldsymbol{\mathcal{K}}(\Omega)$ we have

$$|\langle T_f,\varphi\rangle|\leqslant\mathfrak{p}_{K,0}(\varphi)\int_K|f(x)|\,\mathrm{d}x,$$

with $K$ the support of $\varphi$. Since $\|f\|_{L^1(K)}$ is one of the seminorms defining the topology of $L^1_{\mathrm{loc}}(\Omega)$ the assertion follows.

However, the canonical injection is continuous even when the space $\boldsymbol{\mathcal{K}}'(\Omega)$ is endowed with the **strong-dual topology**. Indeed let $\boldsymbol{\mathcal{B}}$ a bounded subset of $\boldsymbol{\mathcal{K}}(\Omega)$. According to **Proposition 6.58**, there exists a compact subset $K$ of $\Omega$ such that $\sup_{\varphi\in\boldsymbol{\mathcal{B}}}\mathfrak{p}_{K,0}(\varphi)<\infty$. Therefore

$$\sup_{\varphi\in\boldsymbol{\mathcal{B}}}|\langle T_f,\varphi\rangle|\leqslant\left(\sup_{\varphi\in\boldsymbol{\mathcal{B}}}\mathfrak{p}_{K,0}(\varphi)\right)\int_K|f(x)|\,\mathrm{d}x.$$

Since $\|f\|_{L^1(K)}$ is one of the seminorms defining the topology of $L^1_{\mathrm{loc}}(\Omega)$ the assertion follows. ■■■■

## 8.4 Restriction of a distribution, support of a distribution

Let $\Omega$ be an open set of $\mathbb{R}^N$, $U$ an open subset of $\Omega$, and $k\in\bar{\mathbb{N}}$. For any $\varphi\in C_c^k(U)$ the extension by zero of $\varphi$ to $\Omega$, here denoted by $\varphi\cdot\chi_U$, is in $C_c^k(\Omega)$. The map $j\colon\varphi\in C_c^k(U)\to\varphi\cdot\chi_U\in C_c^k(\Omega)$ is clearly **linear** and **injective**. Moreover, its topological counterpart, referred to as the **extension operator**,

$$j\colon\varphi\in\boldsymbol{\mathcal{D}}^k(U)\to\varphi\cdot\chi_U\in\boldsymbol{\mathcal{D}}^k(\Omega),$$

is continuous. Indeed, if $\varphi_n\to 0$ in $\boldsymbol{\mathcal{D}}^k(U)$, then there exists a compact set $K\subseteq U$ such that $\operatorname{supp}\varphi_n\subseteq K$ for every $n\in\mathbb{N}$, and $\mathfrak{p}_{K,k}(\varphi_n)\to 0$. Therefore, also $\mathfrak{p}_{K,k}(\varphi_n\cdot\chi_U)=\mathfrak{p}_{K,k}(\varphi_n)\to 0$, and since $K$ is also a compact subset of $\Omega$, this implies the continuity of $j$.

From the continuity of the extension operator $j\colon \varphi \in \mathcal{D}^k(U) \to \varphi \cdot \chi_U \in \mathcal{D}^k(\Omega)$ and the general theory of **transposition** (cf. Proposition 7.35), it follows that the map

$$j^\top\colon (\mathcal{D}^k)'(\Omega) \to (\mathcal{D}^k)'(U),$$

which maps every distribution $T \in (\mathcal{D}^k)'(\Omega)$ to the (restricted) distribution $T|_U$ in $(\mathcal{D}^k)'(U)$ defined by

$$\langle T|_U, \varphi\rangle = \langle T, j(\varphi)\rangle \quad \textbf{for every } \varphi \in \mathcal{D}^k(U) \tag{8.23}$$

is (linear and) *continuous*. We call the distribution $T|_U \in (\mathcal{D}^k)'(U)$ the **restriction to $U$** of the distribution $T$. When it is clear from the context that $\varphi \in \mathcal{D}^k(U)$, we simply write $\langle T|_U, \varphi\rangle = \langle T, \varphi\rangle$ instead of $\langle T|_U, \varphi\rangle = \langle T, j(\varphi)\rangle$.

### 8.4.1. The Cauchy principal value distribution associated with the pseudofunction $1/x$

For every $\varphi \in \mathcal{D}(\mathbb{R})$ we consider the limit

$$\lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx. \tag{8.24}$$

In general, depending on $\varphi \in \mathcal{D}(\mathbb{R})$, the measurable function $\varphi(x)/x$ can be in $L^1_{\mathrm{loc}}(\mathbb{R})$ or not. This is due to the strong singularity at the origin of the function $x \mapsto 1/x$. Nevertheless, as we will show, for every $\varphi \in \mathcal{D}(\mathbb{R})$ the limit in (8.24) exists. The idea to handle this kind of strong singularities was introduced by Cauchy in his *Mémoire sur les intégrales définies*. Although the measurable function $x \mapsto 1/x$ is not in $L^1_{\mathrm{loc}}(\mathbb{R})$, it belongs to $L^1_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$. In applications, it is quite common to deal with distributions originating from these kinds of functions, and indeed they have a name.

Cauchy, Augustin-Louis. *Mémoire sur les intégrales définies*. Mém. Acad. Sci. Paris 1.82 (1827)

**8.28. Definition.** Let $\Omega$ be an open subset of $\mathbb{R}^N$. A measurable function $f\colon \Omega \to \mathbb{C}$ is called a **pseudofunction** if there exists $x_0 \in \Omega$ such that $f \in L^1_{\mathrm{loc}}(\Omega \setminus \{x_0\})$ although $f \notin L^1_{\mathrm{loc}}(\Omega)$.

To show that (8.24) defines a distribution, we use a first-order Taylor formula with the remainder that, for completeness, we prove below. Recall that a subset $S$ of $\mathbb{R}^N$ is said to be **star-shaped with respect to a point** $x_0 \in \mathbb{R}^N$ if for every $x \in S$ the line segment $[x_0, x]$ from $x_0$ to $x$ lies in $S$. If $S$ is star-shaped with respect to $x_0$, one says that $x_0$ is a *vantage point* of $S$.

**8.29. Theorem. (Brook Taylor (1685-1731))** *Let $\psi \in C^k(\bar{U})$ (with $1 \leqslant k \leqslant \infty$) be a function defined in the closure of a relatively compact open set $\mathbb{R}^N$. Assume that $U$ is* **star-shaped** *with respect to a point $x_0 \in U$. Then, there exists a vector-valued function $\boldsymbol{g}_{x_0} \in C^{k-1}(\bar{U}, \mathbb{R}^N)$, depending on the vantage point $x_0$, such that*

$$\psi(x) = \psi(x_0) + \boldsymbol{g}_{x_0}(x) \cdot (x - x_0) \quad \textbf{and} \quad \boldsymbol{g}_{x_0}(x_0) = \nabla\psi(x_0). \tag{8.25}$$

*Moreover,*

$$\sup_{x \in \bar{U}} |\boldsymbol{g}_{x_0}(x)| \leqslant \sup_{x \in \bar{U}} |\nabla\psi(x)|. \tag{8.26}$$

*Note that, in general, the vector-valued function $\boldsymbol{g}_{x_0}$ coincides with $\nabla\psi$ only at $x_0$.*

**Proof.** Since $U$ is star-shaped with respect to $x_0$, for any $x \in U$ the function

$$t \in [0,1] \mapsto \psi(x_0 + t(x - x_0))$$

is well defined. Moreover, by the fundamental theorem of calculus we have

$$\psi(x) = \psi(x_0) + \left(\int_0^1 \nabla\psi(x_0 + t(x - x_0))\,\mathrm{d}t\right)\cdot(x - x_0) = \psi(x_0) + \boldsymbol{g}_{x_0}(x)\cdot(x - x_0),$$

with

$$\boldsymbol{g}_{x_0}(x) := \int_0^1 \nabla\psi(x_0 + t(x - x_0))\,\mathrm{d}t. \tag{8.27}$$

From (8.27) we infer that $|\boldsymbol{g}_{x_0}(x)| \leqslant \sup_U |\nabla\psi|$ from which (8.26) follows at once. ■

Let us elaborate on the previous limit (8.24). Consider a generic test function $\varphi \in \mathcal{D}(\Omega)$ and set $K := \mathrm{supp}_{\mathbb{R}}\,\varphi$. Take a compact interval $K_a := [-a, a]$ with $a$ sufficiently large to contain the support of $\varphi$. Note that $K_a$ is symmetric with respect to the origin. Also, $K_a$ is star-shaped with respect to the origin of $\mathbb{R}$ and, moreover, since $K \subseteq K_a$ we have

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x = \int_{\{\varepsilon<|x|<a\}} \frac{\varphi(x)}{x}\,\mathrm{d}x.$$

Note that, in general, it is not the case that $K := \mathrm{supp}_{\mathbb{R}}\,\varphi$ is star-shaped because it can be the union of two disjoint intervals. We now use of Taylor's theorem (cf. **Theorem 8.29**). Precisely, for any $x \in K_a$ we can write $\varphi(x) = \varphi(0) + g_0(x)x$ with $g_0(0) = \varphi'(0)$ and $\sup_{x\in K_a} |g_0(x)| \leqslant \sup_{x\in K} |\varphi'(x)|$. Therefore, for $\varepsilon$ sufficiently small we have

$$\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x = \lim_{\varepsilon\to 0}\left(\int_{\{\varepsilon<|x|<a\}} \frac{\varphi(0)}{x}\,\mathrm{d}x + \int_{\{\varepsilon<|x|<a\}} g_0(x)\,\mathrm{d}x\right) \tag{8.28}$$

$$= \int_{K_a} g_0(x)\,\mathrm{d}x. \tag{8.29}$$

Indeed, the first integral in (8.28) is zero because the function $1/x$ is odd. Overall, the map

$$\left\langle \mathrm{vp}\,\frac{1}{x}, \cdot \right\rangle : \varphi \in \mathcal{D}(\Omega) \mapsto \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x \tag{8.30}$$

is well defined and linear. Moreover, since $\left|\int_{K_a} g_0(x)\,\mathrm{d}x\right| \leqslant |K_a|\,\mathfrak{p}_{K,1}(\varphi)$, from (8.29) we infer that

$$\left|\left\langle \mathrm{vp}\,\frac{1}{x}, \cdot \right\rangle\right| \leqslant c_K\,\mathfrak{p}_{K,1}(\varphi), \tag{8.31}$$

for any $\varphi \in \mathcal{D}_K(\mathbb{R})$, with the constant $c_K := |K_a| = 2a$ depending on $K$ only. Thus, the linear functional $\mathrm{vp}\,\frac{1}{x}$ is continuous on every $\mathcal{D}_K(\mathbb{R})$, i.e., continuous on $\mathcal{D}(\mathbb{R})$. By the arbitrariness of $K \in \mathfrak{K}_\Omega$, we get that $\mathrm{vp}\,\frac{1}{x}$ is a **distribution on** $\mathbb{R}$ of order less than or equal to one. It is called the **Cauchy principal value of** $1/x$. By definition,

$$\left\langle \mathrm{vp}\,\frac{1}{x}, \varphi \right\rangle := \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x.$$

After that, let us denote by $U := \mathbb{R}\setminus\{0\}$ the real line pointed at the origin. The function $x \mapsto 1/x$ is locally integrable on $U$ and, therefore, it defines a regular distribution $[1/x]_U$ on $U$. It is simple to show that the *restriction* of $\mathrm{vp}(1/x)$ to $U$ coincides with $[1/x]_U$. Note that $[1/x]_U$ is a Radon measure on $U$ while the extension $\mathrm{vp}(1/x)$ (which is just one of the possible extensions) is not a Radon measure on $\mathbb{R}$. Indeed, while estimate (8.31) only proves that $\mathrm{vp}(1/x)$ is a distribution of order less than or equal to 1, i.e., an element of $\mathrm{vp}\,\frac{1}{x} \in (\mathcal{D}^1)'(\Omega)$, with a simple argument one can show that it cannot be of order zero.

### 8.4.2. Domain of nullity of a distribution

The definition of *domain of nullity* of a distribution mimics the one given for continuous functions in Definition 6.15.

**8.30. Definition.** Let $\Omega$ be an open subset of $\mathbb{R}^N$. We say that a distribution $T \in \mathcal{D}'(\Omega)$ is **null** on the open subset $U \subseteq \Omega$ if its restriction to $U$ is null. In other words, $T$ is null on $U$ if (cf. (8.23))

$$\langle T|_U, \varphi\rangle = 0 \quad \textbf{for all } \varphi \in \mathcal{D}(U).$$

We then call **domain of nullity** of $T \in \mathcal{D}'(\Omega)$, and we denote it by $U_\Omega(T)$, the biggest open subset of $\Omega$ on which $T$ is null (biggest with respect to the set-inclusion order relation). If the restriction of $T$ to any open subset of $U$ is never identically zero, we set $U_\Omega(T) = \emptyset$.

**8.31. Remark.** (On the existence of the domain of nullity) Note that, in principle, the definition of domain of nullity could not be well-posed. Indeed, if we denote by $\tau$ the family of open subsets of $\Omega$, then, formally, we defined $U_\Omega(T)$ as

$$U_\Omega(T) := \max_{\subseteq} \{U \in \tau :: \langle T, \varphi\rangle = 0\, \forall \varphi \in \mathcal{D}(U)\}$$

with the understanding (cf. (8.23)) that $\langle T, \varphi\rangle = \langle T|_U, \varphi\rangle = \langle T, j(\varphi)\rangle$. However, it is not clear if such a maximum exits[8.2] unless one proves that if $T$ is null on every element of a family of open sets, then $T$ is also null on their union. In other words, do not know yet if from the fact that $T|_{U_\lambda}=0$ for every open set of a family $(U_\lambda)_{\lambda\in\Lambda}$ it follows that $T|_U=0$ with $U := \bigcup_{\lambda\in\Lambda} U_\lambda$. The affirmative answer to this question is the object of the **localization principle** below.

**8.32. Remark.** Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, and let $\varphi \in C_c^\infty(\Omega)$. If $g \in C^\infty(\Omega)$ then the set of points where $\varphi \cdot g$ is different from is included in $\operatorname{supp}_\Omega \varphi$ as well as in $\operatorname{supp}_\Omega g$. Therefore,

$$\operatorname{supp}_\Omega(\varphi \cdot g) \subseteq \operatorname{supp}_\Omega \varphi \cap \operatorname{supp}_\Omega g \quad \forall \varphi \in C_c^\infty(\Omega), \forall g \in C^\infty(\Omega).$$

Now, since $\operatorname{supp}_\Omega g$ is closed in $\Omega$, and $\operatorname{supp}_\Omega \varphi$ is a compact subset of $\Omega$, it follows that $\operatorname{supp}_\Omega \varphi \cap \operatorname{supp}_\Omega g$ is compact, and this implies that also $\operatorname{supp}_\Omega(\varphi \cdot g)$ is compact (being a closed subset of a compact subset). In particular, $\varphi \cdot g \in C_c^\infty(\Omega)$ and $\operatorname{supp}_\Omega(\varphi \cdot g) \subseteq \operatorname{supp}_\Omega g$.

After that, suppose that $(g_\lambda)_{\lambda\in\Lambda}$ is a family of functions in $C^\infty(\Omega)$ such that $\operatorname{supp}_\Omega g_\lambda \subseteq U_\lambda$, with $(U_\lambda)_{\lambda\in\Lambda}$ a family of open subsets of $\Omega$. By the previous reasoning, we have that for every $\lambda \in \Lambda$ the function $\varphi \cdot g_\lambda \in C_c^\infty(U_\lambda)$. In particular, $\operatorname{supp}_\Omega(\varphi \cdot g_\lambda)$ is compact and included in $U_\lambda$.

Note that, what we just stated also holds if we assume, more generally, that $\Omega$ is a Hausdorff separated topological space, $\varphi \in C_c(\Omega)$ and $g_\lambda \in C(\Omega)$ with $g_\lambda$ having support (not necessarily compact in $\Omega$) contained in the open subset $U_\lambda \subseteq \Omega$.

**8.33. Lemma.** (Localization principle, I) *Let $\Omega$ be an open subset of $\mathbb{R}^N$, $(U_\lambda)_{\lambda\in\Lambda}$ a family of open subsets of $\Omega$. If the distribution $T$ on $\Omega$ is null on every open subset $U_\lambda \subseteq \Omega$ $(\lambda \in \Lambda)$, then $T$ is null on their union $U := \bigcup_{\lambda\in\Lambda} U_\lambda$.*

**Proof.** We have to show that $\langle T, \varphi\rangle = 0$ for every $\varphi \in \mathcal{D}(U)$. For that, let $(g_\lambda)_{\lambda\in\Lambda}$ be a $C^\infty$ partition of unity of $U$ subordinated to the open cover $(U_\lambda)_{\lambda\in\Lambda}$. The partition of unity $(g_\lambda)_{\lambda\in\Lambda}$ induces a Dieudonné decomposition of every $\varphi \in \mathcal{D}(U)$. Precisely, for any $\lambda \in \Lambda$ we set $\varphi_\lambda :=$

8.2. For example, if $\Omega$ is an open set of $\mathbb{R}^N$, then there does not exist the maximum closed set contained in $\Omega$. While if $K$ is a closed subset of $\mathbb{R}^N$, then there exist the maximum open set contained in $K$. This is because while the property of being an open set is preserved by arbitrary unions, this is not the case for closed sets.

$g_\lambda\varphi$ so that $\operatorname{supp}_\Omega g_\lambda$ is compact and included in $U_\lambda \cap \operatorname{supp}_U \varphi$ (cf. **Remark 8.32**). Next, recall (cf. **Theorem ?**) that for any compact subset $K \in \mathfrak{K}_U$ there exist $n(K) \in \mathbb{N}$ and $\Lambda_{n(K)} := \{\lambda_1, \lambda_2, \ldots, \lambda_{n(K)}\} \subseteq \Lambda$ such that

$$\sum_{\lambda\in\Lambda} g_\lambda = \sum_{\lambda\in\Lambda_{n(K)}} g_\lambda \equiv 1 \quad \text{in } K.$$

In particular, for $K := \operatorname{supp}_U \varphi$, we have $\varphi \equiv \varphi_\lambda \equiv 0$ in $U \setminus K$ and $\varphi = \varphi \cdot 1 = \sum_{\lambda\in\Lambda_{n(K)}} \varphi g_\lambda = \sum_{\lambda\in\Lambda_{n(K)}} \varphi_\lambda$ in $K$. Hence, overall,

$$\varphi(x) = \sum_{\lambda\in\Lambda} \varphi_\lambda(x) = \sum_{\lambda\in\Lambda_{n(K)}} \varphi_\lambda(x) \quad \textbf{for every } x \in U.$$

By the linearity of $T$ we then have

$$\langle T, \varphi\rangle = \sum_{\lambda\in\Lambda_{n(K)}} \langle T, \varphi_\lambda\rangle = 0.$$

By the arbitrariness of $\varphi \in \mathcal{D}(U)$ we get $\langle T, \varphi\rangle$ for every $\varphi \in \mathcal{D}(U)$. This concludes the proof. ■■■■

**8.34. Remark.** When $\Omega$ is a $\sigma$-locally compact Hausdorff space, the localization principle can be proved in the context of Radon measures. However, since open subsets of $\sigma$-locally compact spaces are not necessarily $\sigma$-compact[8.3] (in fact, not necessarily paracompact) one has to be more careful at one point. Precisely, since $U := \bigcup_{\lambda\in\Lambda} U_\lambda$ is an open subset of the $\sigma$-locally compact Hausdorff space $\Omega$, we cannot consider a $C^0$ partition of unity of $U$ subordinated to the open cover $(U_\lambda)_{\lambda\in\Lambda}$. We have to use a simple trick. For every $\varphi \in \mathcal{K}(U)$ we set $K := \operatorname{supp}_U \varphi$ and we complete the open covering $(U_\lambda)_{\lambda\in\Lambda}$ of $U$ to an open covering $(V_\alpha)_{\alpha\in\Gamma}$ of $\Omega$ by adding the open set $\Omega \setminus K$ to $(U_\lambda)_{\lambda\in\Lambda}$. Now we are entitled to consider a $C^0$ partition of unity $(g_\alpha)_{\alpha\in\Gamma}$ of $\Omega$ subordinated to the open cover $(V_\alpha)_{\alpha\in\Gamma}$. For any $\alpha \in \Gamma$ we set $\varphi_\alpha := g_\alpha \varphi$ so that $\operatorname{supp}_\Omega g_\alpha$ is compact and included in $V_\alpha \cap K$. After that, there exist $n(K) \in \mathbb{N}$ and $\Gamma_{n(K)} := \{\alpha_1, \alpha_2, \ldots, \alpha_{n(K)}\} \subseteq \Gamma$ such that

$$\sum_{\alpha\in\Gamma} g_\alpha = \sum_{\alpha\in\Lambda_{n(K)}} g_\alpha \equiv 1 \quad \text{in } K.$$

Moreover, since $\varphi \equiv \varphi_\alpha \equiv 0$ in $U \setminus K$ and $\varphi = \varphi \cdot 1 = \sum_{\alpha\in\Lambda_{n(K)}} \varphi g_\alpha = \sum_{\alpha\in\Lambda_{n(K)}} \varphi_\alpha$ in $K$, we have that

$$\varphi(x) = \sum_{\alpha\in\Gamma} \varphi_\alpha(x) = \sum_{\alpha\in\Lambda_{n(K)}} \varphi_\alpha(x) \quad \textbf{for every } x \in U.$$

By the linearity of $T$ we then have $\langle T, \varphi\rangle = \sum_{\alpha\in\Lambda_{n(K)}} \langle T, \varphi_\alpha\rangle = 0$, and by the arbitrariness of $\varphi \in \mathcal{K}(U)$ we get $\langle T, \varphi\rangle$ for every $\varphi \in \mathcal{K}(U)$. This concludes the proof. ⋯

By the localization principle, we obtain the existence of the domain of nullity of a distribution.

**8.35. Corollary.** (DOMAIN OF NULLITY) *The domain of nullity $U_\Omega(T)$ of a distribution $T$ exists and is unique.*

8.3. However, in metric spaces, open subsets of $\sigma$-locally compact space are still $\sigma$-compact. Indeed, in a metric space every open set is an $F_\sigma$ (because every closed set $F$ is a $G_\delta$: $F = \cap_j A_j$ where $A_j = \{x \in \Omega :: d(x, F) < \frac{1}{j}\}$). Thus, for every open subset $A$ of $\Omega$, there exists a sequence $(F_j)_{j\in\mathbb{N}}$ of closed subsets of $\Omega$ such that a $A = \cup_{j\in\mathbb{N}} F_j$. But each $F_j$ is $\sigma$-compact (because this is a weakly hereditary property) and, therefore, $A$ can be expressed as a countable union of countably many compact subsets.

In general, without some extra condition on $\Omega$ (like the metrizability condition), the result does not hold.

**Proof.** The existence is a consequence of the localization principle because if $T$ is null on a family of open subsets of $\Omega$ then $T$ is null also on their union. After that, uniqueness is a consequence of the fact that any maximum (here with respect to the set inclusion order relation) is unique.

**8.36. Corollary.** (Localization principle, II) *Let $\Omega$ be an open subset of $\mathbb{R}^N$, $(U_\lambda)_{\lambda\in\Lambda}$ a family of open subsets of $\Omega$. Let $T_1, T_2$ be two distributions on $\Omega$ such that $T_1|_{U_\lambda} \equiv T_2|_{U_\lambda}$ for every $\lambda \in \Lambda$, then $T_1 \equiv T_2$ on the union $U := \bigcup_{\lambda\in\Lambda} U_\lambda$, i.e., $T_1|_U \equiv T_2|_U$.*

**Proof.** It is sufficient to note that, by hypotheses, $(T_1 - T_2)|_{U_\lambda} \equiv 0$ for every $\lambda \in \Lambda$, so that, by the localization principle, $(T_1 - T_2)|_U \equiv 0$.

**8.37. Remark.** The same results hold for Radon measures on $\sigma$-locally compact Hausdorff spaces (cf. **Remark 8.34**).

### 8.4.3. The support of a distribution

The definition of support of a distribution mimics the one given for continuous functions in **Definition 6.15**.

**8.38. Definition.** We call **support** of a distribution $T \in \mathcal{D}'(\Omega)$ the complement, relative to $\Omega$, of its domain of nullity: $\operatorname{supp}_\Omega T := \Omega \setminus U_\Omega(T)$. It is clear that $\operatorname{supp}_\Omega T$ is closed in the relative topology of $\Omega$.

**Example 8.39.** (Dirac $\delta$ distribution) If $\delta_a$ is the Dirac distribution centered at $a \in \Omega \subseteq \mathbb{R}^N$, then $\operatorname{supp} = \{a\}$.

**Example 8.40.** (Regular distributions) If $f \in L^1_{\text{loc}}(\Omega)$ and $T_f \in \mathcal{D}'(\Omega)$ is the regular distribution associated with $f$, then $\operatorname{supp}_\Omega T_f$ coincides with the *essential support* of $f$, denoted by $\operatorname{ess\,supp}_\Omega f$. We recall that the essential support of a function in $L^1_{\text{loc}}(\Omega)$ is the complement of the *essential domain of nullity* $\operatorname{ess} U_\Omega(f)$of $f$, which is defined as the biggest open subset of $\Omega$ on which $f$ is a.e. equal to zero. Note that the existence of such a biggest open subset (i.e., of $\operatorname{ess} U_\Omega(f)$) is guaranteed by the localization principle (**Lemma 8.33**). Indeed, if $(U_\lambda)_{\lambda\in\Lambda}$ is a family of open subsets of $\Omega$ such that $T_f|_{U_\lambda} \equiv 0$ then, by **Theorem 8.24**, we know that $f$ is a.e. equal to zero on each $U_\lambda$, but we cannot infer from this that $f \equiv 0$ a.e. in $U := \bigcup_{\lambda\in\Lambda} U_\lambda$ (because $\Lambda$ can be, in terms of cardinality, more than countable). However, as already said, the localization principle assures that this indeed the case.

For completeness, we also present a different argument that shows $\operatorname{ess} U_\Omega(f)$ is well-defined. The key is that $\mathbb{R}^N$ (and therefore $\Omega$) is a second-countable space. Indeed, let $(B_n)_{n\in\mathbb{N}}$ be a countable base of $\Omega$, and denote by $(U_n)_{n\in\mathbb{N}}$ the subsequence of $(B_n)_{n\in\mathbb{N}}$ consisting of those basis sets such that $f|_{U_n} \equiv 0$ a.e. in $U_n$. We want to show that $U_\Omega f = \bigcup_{n\in\mathbb{N}} U_n$. For that, we observe that $\bigcup_{n\in\mathbb{N}} U_n$ is an open set and $f \equiv 0$ a.e. on $\bigcup_{n\in\mathbb{N}} U_n$ because $\bigcup_{n\in\mathbb{N}} U_n$ is a countable union of open sets. Moreover, $\bigcup_{n\in\mathbb{N}} U_n$ is the biggest open set where $f$ is a.e. equal to zero because if $V$ is any other open set where $f$ is a.e. equal to zero, then necessarily $V \subseteq \bigcup_{n\in\mathbb{N}} U_n$ as $V$ would be expressible as a countable union of elements of $(B_n)_{n\in\mathbb{N}}$ where $f$ is a.e. equal to zero.

**8.41. Remark.** Note that the notion of essential support is the natural notion of support in Lebesgue spaces. Indeed, if $\chi_{\mathbb{Q}} \in L^1_{\text{loc}}(\mathbb{R})$ is the indicator function of the rationals, and $T_{\chi_{\mathbb{Q}}}$ the corresponding regular distribution, then $\operatorname{supp}_{\mathbb{R}} \chi_{\mathbb{Q}} = \mathbb{R}$ (because $U_{\mathbb{R}}(\chi_{\mathbb{Q}}) = \emptyset$) whereas $\operatorname{ess\,supp}_\Omega \chi_{\mathbb{Q}} = \operatorname{supp}_\Omega T_{\chi_{\mathbb{Q}}} = \emptyset$ (because $\operatorname{ess} U_\Omega(f) = U_\Omega(T_f) = \mathbb{R}$). In fact, the regular distribution $T_{\chi_{\mathbb{Q}}}$ is nothing but the null functional.

**Example 8.42.** (CONTINUOUS FUNCTIONS) If $f \in C(\Omega) \cap L^1_{\text{loc}}(\mathbb{R})$ then

$$\operatorname{supp}_\Omega f = \operatorname{ess\,supp}_\Omega f = \operatorname{supp}_\Omega T_f.$$

In other words, for a regular distribution that admits a continuous representative, all three notions of support are the same. ⋯

**Example 8.43.** (RADON MEASURES) f $\mu$ is a Radon measure on the $\sigma$-locally compact Hausdorff space $\Omega$, we can define its restriction to an open subset $U$ of $\Omega$ by setting

$$\langle \mu|_U, \varphi \rangle = \langle \mu, \varphi \rangle \quad \forall \varphi \in \boldsymbol{\mathcal{K}}(U).$$

Therefore, one can also define the support and the domain of nullity of a Radon measure in exactly the same way we did for elements of $\boldsymbol{\mathcal{D}}'(\Omega)$. If $\Omega$ is an open subset of $\mathbb{R}^N$, then it is simple to check that the distribution associated with $\mu|_U$ coincides with the restriction to $U$ of the distribution associated with $\mu$ (to see this one uses the density of $C_c^\infty(\Omega)$ in $\boldsymbol{\mathcal{K}}(U)$). Therefore, when $\Omega \subseteq \mathbb{R}^N$, the support of a Radon measure coincides with the support of the corresponding distribution (cf. the **Consistency Theorem 7.41**) ⋯

An immediate consequence of the definition of support is stated in the following result.

**8.44. Proposition.** *If $\varphi \in \boldsymbol{\mathcal{D}}(\Omega), T \in \boldsymbol{\mathcal{D}}'(\Omega)$ and*

$$\operatorname{supp}_\Omega \varphi \cap \operatorname{supp}_\Omega T = \emptyset,$$

*then*

$$\langle T, \varphi \rangle = 0.$$

*In particular, if $\varphi$ is zero* **in a neighborhood** *of $\operatorname{supp}_\Omega T$ then $\langle T, \varphi \rangle = 0$.*

Note that the condition $\varphi \equiv 0$ on $\operatorname{supp}_\Omega T$ is not sufficient, in general, to conclude that $\langle T, \varphi \rangle = 0$. For example, if $T : \varphi \in \boldsymbol{\mathcal{D}}(\mathbb{R}) \mapsto \varphi'(0)$ then

$$\operatorname{supp}_\mathbb{R} T = \{0\}.$$

On the other hand, if $\varphi \in \boldsymbol{\mathcal{D}}(\mathbb{R})$ is such that $\varphi(x) \equiv x$ in a neighborhood of zero, then $\varphi \equiv 0$ on $\operatorname{supp}_\mathbb{R} T$ (i.e., $\varphi(0) = 0$) although $\langle T, \varphi \rangle = \varphi'(0) = 1 \neq 0$.

Roughly speaking, the reason why the previous counterexample works is that the validity of the statement $\langle T, \varphi \rangle = 0$ whenever $\varphi \equiv 0$ on $\operatorname{supp}_\Omega T$, depends on the order of the distribution $T$ with respect to the order of the zeros where $\varphi \equiv 0$ on $\operatorname{supp}_\Omega T$. In fact, later on, we will show the following fundamental result.

**8.45. Proposition.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$.*

**Case $m \in \mathbb{N}$.**

*Let $T \in (\boldsymbol{\mathcal{D}}^m)'(\Omega)$, $\varphi \in \boldsymbol{\mathcal{D}}^m(\Omega)$.* **Assume** *that $\varphi \in \boldsymbol{\mathcal{D}}^m(\Omega)$ is such that $(\partial^\beta \varphi)(x) = 0$ for any $x \in \operatorname{supp}_\Omega T$ and for every multi-index $\beta \in \mathbb{N}^N$ having modulus $|\beta| \leqslant m$,* **then** *$\langle T, \varphi \rangle = 0$.*

**Case $m = \infty$.**

*Let $T \in \boldsymbol{\mathcal{D}}'(\Omega)$, $\varphi \in \boldsymbol{\mathcal{D}}(\Omega)$.* **Assume** *that $(\partial^\beta \varphi)(x) = 0$ for any $x \in \operatorname{supp}_\Omega T$ and for every multi-index $\beta \in \mathbb{N}^N$,* **then** *$\langle T, \varphi \rangle = 0$.*

## 8.5 Principe du recollement des morceaux. Gluing lemma.

The next result allows for a passage from locally defined distributions to globally defined ones. This is achieved via a gluing procedure referred to as *principe du recollement des morceaux* or gluing lemma. The french name is the one used by L. Schwartz in his treatise on the theory of distributions.

**8.46. Theorem.** (PRINCIPE DU RECOLLEMENT DES MORCEAUX) *Let $\Omega$ be an open subset of $\mathbb{R}^N$ and $(\Omega_\lambda)_{\lambda\in\Lambda}$ an open cover of $\Omega$. For every index $\lambda\in\Lambda$ it is given a distribution $T_\lambda\in\mathcal{D}'(\Omega_\lambda)$.*

**Assume** *that the family $(T_\lambda)_{\lambda\in\Lambda}$ satisfies the following condition:*

*For every $\lambda,\mu\in\Lambda$ with $\Omega_\lambda\cap\Omega_\mu\neq\emptyset$, the restrictions of $T_\lambda$ and $T_\mu$ to the open set $\Omega_\lambda\cap\Omega_\mu$ coincide. In other words, suppose that $T_\lambda|_{\Omega_\lambda\cap\Omega_\mu}\equiv T_\mu|_{\Omega_\lambda\cap\Omega_\mu}$ for every $\lambda,\mu\in\Lambda$, i.e.,*

$$\langle T_\lambda,\varphi\rangle=\langle T_\mu,\varphi\rangle\quad\forall\varphi\in\mathcal{D}(\Omega_\lambda\cap\Omega_\mu).$$

**Claim:** *There exists, and is unique, the distribution $T\in\mathcal{D}'(\Omega)$ such that the restriction of $T$ to every $\Omega_\lambda$ coincides with $T_\lambda$, i.e., such that $T|_{\Omega_\lambda}\equiv T_\lambda$ for every $\lambda\in\Lambda$, i.e., such that*

$$\langle T,\varphi\rangle=\langle T_\lambda,\varphi\rangle\quad\forall\varphi\in\mathcal{D}(\Omega_\lambda).$$

**Moreover:**

*i. If every $T_\lambda$ is of order less than or equal to $k\in\mathbb{N}$, then $T$ as well is of order less than or equal to $k$.*

*ii. If every $T_\lambda$ is a regular distribution, then also $T$ is a regular distribution.*

*iii. If every $T_\lambda$ is $C^k(\Omega)$, then also $T$ is in $C^k(\Omega)$.*

PROOF. The niqueness of $T$ follows from the **localization principle** (cf. Corollary 8.36). Indeed, if $T$ and $S$ are such that $T|_{\Omega_\lambda}\equiv T_\lambda$ and $S|_{\Omega_\lambda}\equiv T_\lambda$ then $T|_{\Omega_\lambda}\equiv S|_{\Omega_\lambda}$ for every $\lambda\in\Lambda$. Hence $S\equiv T$.

Let us prove the **existence** of $T$. Let $(g_\lambda)_{\lambda\in\Lambda}$ be a $C^\infty$-partition of unity on $\Omega$, subordinated to the open cover $(\Omega_\lambda)_{\lambda\in\Lambda}$. The partition of unity $(g_\lambda)_{\lambda\in\Lambda}$ induces a Dieudonné decomposition of every $\varphi\in\mathcal{D}(U)$. Precisely, for any $\lambda\in\Lambda$ we set $\varphi_\lambda:=g_\lambda\varphi$. Clearly, one has $\varphi_\lambda\in\mathcal{D}(\Omega_\lambda)$ because $\varphi$ has compact support in $\Omega$ and $\operatorname{supp}_\Omega g_\lambda\subseteq\Omega_\lambda$. Recall (cf. Theorem ?) that for any compact subset $K\in\mathfrak{K}_\Omega$ there exist $n(K)\in\mathbb{N}$ and $\Lambda_{n(K)}:=\{\lambda_1,\lambda_2,\ldots,\lambda_{n(K)}\}\subseteq\Lambda$ such that

$$\sum_{\lambda\in\Lambda}g_\lambda=\sum_{\lambda\in\Lambda_{n(K)}}g_\lambda=1\quad\text{in }K.$$

In particular, for $K:=\operatorname{supp}_\Omega\varphi$, we get

$$\varphi(x)=\sum_{\lambda\in\Lambda_{n(K)}}\varphi_\lambda(x)\ \textbf{ for every }x\in\Omega,\qquad\varphi_\lambda:=g_\lambda\varphi.\tag{8.32}$$

We define the candidate distribution $T$ as follows[8.4]. For any $\varphi\in\mathcal{D}(\Omega)$ we set

$$\langle T,\varphi\rangle:=\sum_{\lambda\in\Lambda}\langle T_\lambda,\varphi_\lambda\rangle=\sum_{\lambda\in\Lambda_{n(K)}}\langle T_\lambda,\varphi_\lambda\rangle=\sum_{\lambda\in\Lambda_{n(K)}}\langle T_\lambda,\varphi g_\lambda\rangle.\tag{8.33}$$

This is a natural ansatz. Indeed, if a globally defined distribution $T$ exists, then it must necessarily

8.4. Note that the definition of the map $T$ can be equivalently stated as $T:\varphi\in\mathcal{D}(\Omega)\mapsto\sum_{\lambda\in\Lambda}\langle T_\lambda,\varphi_\lambda\rangle$. The expression in (8.33) depends on $\varphi$, while the previous one does not. However, in writing (8.33) we are stressing that for any $\varphi\in\mathcal{D}(\Omega)$ the sum $\sum_{\lambda\in\Lambda}\langle T_\lambda,\varphi_\lambda\rangle$ reduces to a finite sum, whose number of nonvanishing terms depend on the support of $\varphi$.

satisfy (8.33).

Note that (8.33) defines a linear form $T$ on $\mathfrak{D}(\Omega)$. Moreover, for any compact subset $K \in \mathfrak{K}_\Omega$ of $\Omega$, the restriction of $T$ to $\mathfrak{D}_K(\Omega)$ is continuous. Indeed, for every $\lambda \in \Lambda$ the map

$$\varphi \in \mathfrak{D}_K(\Omega) \mapsto \varphi_\lambda := g_\lambda \varphi \in \mathfrak{D}(\Omega)$$

is a continuous map from $\mathfrak{D}_K(\Omega)$ to $\mathfrak{D}(\Omega)$ (cf. **Proposition 6.63**). Therefore, also the map $\varphi \in \mathfrak{D}_K(\Omega) \mapsto \langle T_\lambda, \varphi_\lambda \rangle$ is continuous on $\mathfrak{D}_K(\Omega)$ because a composition of two continuous functions. Since $\langle T, \varphi \rangle$ is a finite sum of linear and continuous functionals, $T$ is a linear and continuous functional on $\mathfrak{D}_K(\Omega)$. By the arbitrariness of the compact set $K \in \mathfrak{K}_\Omega$, we infer that $T$ is linear and continuous on $\mathfrak{D}(\Omega)$ and, therefore, a distribution on $\Omega$.

Note that all that we have derived, in principle, depends on the Dieudonné decomposition of $\varphi$. But as we are going to show, the restriction of $T$ to $\Omega_\mu$ coincide with $T_\mu$ for every $\mu \in \Lambda$. After that, the localization principle implies that the distribution $T$ is unique and, therefore, independent from any specific $C^\infty$-partition of unity used to define it.

We have to show that for every $\mu \in \Lambda$ one has

$$\langle T, \varphi \rangle = \langle T_\mu, \varphi \rangle \quad \varphi \in \mathfrak{D}(\Omega_\mu).$$

Let $\varphi \in \mathfrak{D}(\Omega_\mu)$ with $\mu \in \Lambda$. Then $\varphi g_\lambda \in \mathfrak{D}(\Omega_\mu \cap \Omega_\lambda)$ whenever $\Omega_\mu \cap \Omega_\lambda \neq \emptyset$. By hypothesis, if $\Omega_\mu \cap \Omega_\lambda \neq \emptyset$, then $T_\mu$ and $T_\lambda$ coincide on $\Omega_\mu \cap \Omega_\lambda$. In other words,

$$\langle T_\mu, \varphi g_\lambda \rangle = \langle T_\lambda, \varphi g_\lambda \rangle \quad \forall \mu, \lambda \in \Lambda :: \Omega_\mu \cap \Omega_\lambda \neq \emptyset. \tag{8.34}$$

By (8.33) and (8.34) it follows that (we set $K := \operatorname{supp}_\Omega \varphi \subseteq \Omega_\mu$)

$$\langle T, \varphi \rangle \overset{(8.33)}{=} \sum_{\lambda \in \Lambda_{n(K)}} \langle T_\lambda, \varphi g_\lambda \rangle \overset{(8.34)}{=} \sum_{\lambda \in \Lambda_{n(K)}} \langle T_\mu, \varphi g_\lambda \rangle \overset{(8.32)}{=} \left\langle T_\mu, \sum_{\lambda \in \Lambda_{n(K)}} \varphi_\lambda \right\rangle \overset{(8.32)}{=} \langle T_\mu, \varphi \rangle.$$

This concludes the proof for the case in which every $T_\lambda$ is in $\mathfrak{D}'(\Omega_\lambda)$.

*Proof of i.* Now, if every $T_\lambda$ is continuous on $C_c^\infty(\Omega_\lambda)$ for the topology of $\mathfrak{D}^k(\Omega_\lambda)$, it is easy to check, by following exactly the same lines, that $T$ defined by (8.33) is continuous on $C_c^\infty(\Omega)$ for the topology of $\mathfrak{D}^k(\Omega)$.

*Proof of ii.* If $(f_\lambda)_{\lambda \in \Lambda}$ are in $L^1_{\mathrm{loc}}(\Omega_\lambda)$, then, with $T_\lambda := T_{f_\lambda}$ satisfying the hypotheses of the theorem, for every $\varphi \in \mathfrak{D}(\Omega)$ we have (with $K := \operatorname{supp}_\Omega \varphi$)

$$\begin{aligned}
\langle T, \varphi \rangle \overset{(8.33)}{=} & \sum_{\lambda \in \Lambda_{n(K)}} \langle T_\lambda, \varphi_\lambda \rangle = \sum_{\lambda \in \Lambda_{n(K)}} \int_{\Omega_\lambda} f_\lambda(x) \varphi_\lambda(x)\, \mathrm{d}x \\
= & \sum_{\lambda \in \Lambda_{n(K)}} \int_\Omega f_\lambda(x) g_\lambda(x) \varphi(x)\, \mathrm{d}x = \int_\Omega \left( \sum_{\lambda \in \Lambda_{n(K)}} f_\lambda(x) g_\lambda(x) \right) \varphi(x)\, \mathrm{d}x \\
= & \int_\Omega \left( \sum_{\lambda \in \Lambda} f_\lambda(x) g_\lambda(x) \right) \varphi(x)\, \mathrm{d}x.
\end{aligned}$$

Note that, since $\operatorname{supp}_\Omega \varphi_\lambda \subseteq \Omega_\lambda$, one can replace the integrals on $\Omega_\lambda$ with integrals extended to the whole of $\Omega$. Also, it is important to write down the last equality because we want the term in parenthesis to not depend on the test function. After that, the unique distribution $T$ obtained via gluing, is nothing but the regular distribution associated with

$$f(x) := \sum_{\lambda \in \Lambda} f_\lambda(x) g_\lambda(x), \quad x \in \Omega.$$

*Proof of iii.* Finally, if $(f_\lambda)_{\lambda\in\Lambda}$ is a family in $C^k(\Omega)$ then $f(x)=\sum_{\lambda\in\Lambda} f_\lambda(x)g_\lambda(x)$ is in $C^k(\Omega)$ because for every $x\in\Omega$ the previous sum consists of finitely many terms.

**Example 8.47.** (Application to Sobolev spaces) Let us assume that $\Omega$ is an open subset of $\mathbb{R}^N$ (bounded or not). The Sobolev space $W^{k,p}_{\text{loc}}(\Omega)$ is defined by

$$W^{k,p}_{\text{loc}}(\Omega)=\{u\in L^p_{\text{loc}}(\Omega) :: u|_U\in W^{k,p}(U) \text{ for all } U\Subset\Omega\}.$$

A sequence $(u_n)_{n\in\mathbb{N}}$ converges to $u$ in $W^{k,p}_{\text{loc}}(\Omega)$ if for every $U\Subset\Omega$ one has $\|u_n-u\|_{W^{k,p}(U)}\to 0$ when $n\to\infty$.

Let $(\Omega_\lambda)_{\lambda\in\Lambda}$ be an open cover of $\Omega$ made by **bounded sets**. Let us assume that $u_\lambda\in W^{k,p}(\Omega_\lambda)$ for every $\lambda\in\Lambda$ and that if $\Omega_\lambda\cap\Omega_\mu\neq\emptyset$ then $u_\lambda\equiv u_\mu$ on the intersection. Then there exists a unique regular distribution $u\in W^{k,p}_{\text{loc}}(\Omega)$ such that the restriction of $u$ to every $\Omega_\lambda$ coincides with $u_\lambda$. In fact,

$$u(x)=\sum_{\lambda\in\Lambda} u_\lambda(x)g_\lambda(x), \quad x\in\Omega$$

with $(g_\lambda)_{\lambda\in\Lambda}$ an arbitrary $C^\infty$-partition of unity on $\Omega$ subordinated to the open cover $(\Omega_\lambda)_{\lambda\in\Lambda}$.

**Example 8.48.** (Simple and double layer distributions) Let $\Sigma$ be a $C^1$-hypersurface of $\mathbb{R}^N$. Let $(\Omega_i,\Sigma_i,\Phi_i)_{i\in I}$ an atlas of $\Sigma$ and $\sigma_i$ the positive Radon measure induced on $\Sigma_i$ by the Lebesgue measure on $\Omega_i\subseteq\mathbb{R}^{N-1}$. It can be sown that if $\Sigma_i\cap\Sigma_j\neq\emptyset$ the restrictions of $\sigma_i$ and $\sigma_j$ to $\Sigma_i\cap\Sigma_j$ coincide. By the *principe du recollement des morceaux* there exists a unique positive Radon measure $\sigma$ on $\Sigma$ whose restriction to $\Sigma_i$ coincide with $\sigma_i$. It is possible to show that $\sigma$ does not depend on the chosen atlas on $\Sigma$. The measure $\sigma$ so built is called the measure induced on $\Sigma$ from the Lebesgue measure on $\mathbb{R}^{N-1}$. If $\sigma$ is a bounded measure, then $\sigma(1)$ is called the **area** of $\Sigma$. For any $\sigma$-integrable function $\rho$ one sets

$$\sigma(\rho):=\int_\Sigma \rho\,\mathrm{d}\sigma.$$

After that, for any $\sigma$-integrable function $\rho$ the linear form

$$\varphi\in\boldsymbol{\mathcal{K}}(\mathbb{R}^N)\mapsto\sigma(\varphi\rho)=\int_\Sigma \varphi\,\rho\,\mathrm{d}\sigma$$

is continuous on $\boldsymbol{\mathcal{K}}(\mathbb{R}^N)$. Therefore, this form is a distribution on $\mathbb{R}^N$ and is called **distribution of simple layer**, of density $\rho$, and concentrated on $\Sigma$. It is simple to show that such a distribution is a Radon measure whose support is contained in $\Sigma$. If the hypersurface $\Sigma$ is oriented by the unit normal field $\nu$, then the linear form

$$\varphi\in\boldsymbol{\mathcal{D}}^1(\mathbb{R}^N)\mapsto\sigma(\varphi\rho)\int_\Sigma \partial_\nu\varphi\,\rho\,\mathrm{d}\sigma$$

is continuous on $\boldsymbol{\mathcal{D}}^1(\mathbb{R}^N)$. Therefore, this form is a distribution of order less than or equal to $1$ and is called the **distribution of double layer**, of density $\rho$, and concentrated on $\Sigma$.

# 9

# Derivatives of Distributions

## 9.1 | Definition and first consequences

**9.1. Definition.** Let $\Omega$ be an open subset of $\mathbb{R}^N$. Let $T \in \boldsymbol{\mathcal{D}}'(\Omega)$ and $\alpha \in \mathbb{N}^N$ a multi-index. We define the **partial derivative of index** $\alpha$ of the distribution $T$ as the map

$$\boldsymbol{\mathcal{D}}(\Omega) \ni \varphi \mapsto \langle \mathsf{D}^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, \mathsf{D}^\alpha \varphi \rangle.$$

We collect in the next proposition the first main properties of the derivative operator.

**9.2. Proposition.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$. Let $T \in \boldsymbol{\mathcal{D}}'(\Omega)$ and $\alpha \in \mathbb{N}^N$ a multi-index. The following assertions hold:*

*i.* $\mathsf{D}^\alpha T$ *is a distribution on $\Omega$ and the map $\mathsf{D}^\alpha : T \mapsto \mathsf{D}^\alpha T$ is linear and continuous from $\boldsymbol{\mathcal{D}}'(\Omega)$ into $\boldsymbol{\mathcal{D}}'(\Omega)$ (no matter if we endow the space of distributions with the weak-dual topology or with the strong-dual topology). More precisely, for any $k \in \bar{\mathbb{N}}$ the map $\mathsf{D}^\alpha$ is a linear and continuous operator from $(\boldsymbol{\mathcal{D}}^k)'(\Omega)$ into $(\boldsymbol{\mathcal{D}}^{k+|\alpha|})'(\Omega)$.*

*ii. If $S := T|_U$ is the restriction of $T$ to an open subset $U$ of $\Omega$ then for any multi-index $\alpha \in \mathbb{N}^N$ the distributions $\mathsf{D}^\alpha S$ is the restriction of $\mathsf{D}^\alpha T$ to $U$. In symbols:*

$$\mathsf{D}^\alpha (T|_U) \equiv (\mathsf{D}^\alpha T)|_U.$$

*It follows that*

$$\operatorname{supp}_\Omega \mathsf{D}^\alpha T \subseteq \operatorname{supp}_\Omega T.$$

*iii. For multi-indices $\alpha$ and $\beta$ in $\mathbb{N}^N$ the theorem on the symmetry of partial derivatives holds:*

$$\mathsf{D}^{\alpha+\beta} T = \mathsf{D}^\alpha \mathsf{D}^\beta T = \mathsf{D}^\beta \mathsf{D}^\alpha T \quad \forall T \in \boldsymbol{\mathcal{D}}'(\Omega).$$

*iv. For any $T \in \boldsymbol{\mathcal{D}}'(\Omega)$ and $f \in \boldsymbol{\mathcal{E}}(\Omega)$ one has the Leibniz formula*

$$\mathsf{D}^\alpha (fT) = \sum_{\beta \leqslant \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \mathsf{D}^\beta f \mathsf{D}^{\alpha-\beta} T.$$

**9.3. Corollary.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$. Let $(T_j)_{j \in \mathbb{N}}$ a sequence of distributions.* **Assume** *that the series $\sum_{j \in \mathbb{N}} T_j$ converges towards a distribution $T \in \boldsymbol{\mathcal{D}}'(\Omega)$ with respect to the weak-dual topology* *(resp. the strong-dual topology)* **then** *for any multi-index* $\alpha \in \mathbb{N}^N$ *the series* $\sum_{j\in\mathbb{N}} \mathsf{D}^\alpha T_j$ *converge towards* $\mathsf{D}^\alpha T \in \mathcal{D}'(\Omega)$ *in the weak-dual topology (resp. in the strong-dual topology). In symbols*

$$\sum_{j\in\mathbb{N}} T_j \to T \ \ \textbf{in } \mathcal{D}'_\sigma(\Omega) \ \Rightarrow \sum_{j\in\mathbb{N}} \mathsf{D}^\alpha T_j \to \mathsf{D}^\alpha T \ \ \textbf{in } \mathcal{D}'_\sigma(\Omega)$$

*and*

$$\sum_{j\in\mathbb{N}} T_j \to T \ \ \textbf{in } \mathcal{D}'_\flat(\Omega) \ \Rightarrow \sum_{j\in\mathbb{N}} \mathsf{D}^\alpha T_j \to \mathsf{D}^\alpha T \ \ \textbf{in } \mathcal{D}'_\flat(\Omega)$$

*where the subscripts* $\sigma$ *and* $\flat$ *stands for the weak and strong dual topology on* $\mathcal{D}'(\Omega)$.

**PROOF.** From the linearity and the (sequentially) continuity of the map operator $\mathsf{D}^\alpha$ proved in Proposition 9.2 at comma **(i)**, we have that $T = \lim_{i\to\infty} \sum_{j\in\mathbb{N}_i} T_j$ and therefore

$$\begin{aligned} \mathsf{D}^\alpha T &= \lim_{i\to\infty}\left(\mathsf{D}^\alpha \sum_{j\in\mathbb{N}_i} T_j\right) \\ &= \lim_{i\to\infty}\left(\sum_{j\in\mathbb{N}_i} \mathsf{D}^\alpha T_j\right) \\ &= \sum_{j\in\mathbb{N}} \mathsf{D}^\alpha T_j \end{aligned}$$

both if the limit is taken with respect to the weak-dual topology or with respect to the strong-dual topology. ■■■■

**Example 9.4.** We denote by $\mathbb{R}^N_+$ the hyperoctant $\{x \equiv (x_1, \ldots, x_N) \in \mathbb{R}^N :: x_i \geqslant 0\}$. The Heaviside function $H: \mathbb{R}^N \to \mathbb{R}$ is nothing but the characteristic function of $\mathbb{R}^N_+$. This is a locally integrable function. We want to compute the gradient of the regular distribution $T_H$. For every $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^N, \mathbb{R}^N)$, we have

$$\begin{aligned} \langle \nabla T_H, \boldsymbol{\varphi}\rangle &:= -\int_{\mathbb{R}^N} H(x)\,\mathrm{div}\boldsymbol{\varphi}(x)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^N_+} \mathrm{div}\boldsymbol{\varphi}(x)\,\mathrm{d}x \\ &= -\int_{\partial\mathbb{R}^N_+} \boldsymbol{\varphi}(\sigma)\cdot\boldsymbol{n}(\sigma)\,\mathrm{d}\sigma. \end{aligned}$$

with $\boldsymbol{n}$ the exterior unit normal to $\partial\mathbb{R}^N_+$. Note that, if we denote by $e_i^\perp := \{x \equiv (x_1, \ldots, x_N) \in \mathbb{R}^N_+ :: x_i = 0\}$ the face of $\mathbb{R}^N_+$ perpendicular to $e_i$, then

$$\langle \nabla T_H, \boldsymbol{\varphi}\rangle = \sum_{i=1}^N \left(\int_{e_i^\perp} \boldsymbol{\varphi}(\sigma)\cdot e_i\,\mathrm{d}\sigma\right). \tag{9.1}$$

Eventually, note that for $N=1$, we obtain $e_1^\perp = \{0\}$ so that for every $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{R})$

$$\langle T'_H, \varphi\rangle = \int_{e_1^\perp} \varphi(\sigma)\,\mathrm{d}\sigma \equiv \varphi(0).$$

Next, let us compute $\frac{\partial^N}{\partial x_1 \cdots \partial x_N} T_H$. We have, for every $\varphi \in \mathcal{D}(\mathbb{R}^N, \mathbb{R})$

$$\left\langle \frac{\partial^N}{\partial x_1 \cdots \partial x_N} T_H, \varphi \right\rangle = \int_{\mathbb{R}^N_+} \frac{\partial^N}{\partial x_1 \cdots \partial x_N} \varphi(x)\,\mathrm{d}x$$

$$\begin{aligned} &= \int_0^{+\infty} \cdots \int_0^{+\infty} \int_0^{+\infty} \frac{\partial^N}{\partial x_1 \cdots \partial x_N} \varphi(x)\, \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_N \\ &= \int_0^{+\infty} \cdots \int_0^{+\infty} \frac{\partial^N}{\partial x_1 \cdots \partial x_N} \varphi(0, x_2, \cdots, x_N)\, \mathrm{d}x_2 \cdots \mathrm{d}x_N \\ &= \varphi(0). \end{aligned}$$

Hence

$$\frac{\partial^N}{\partial x_1 \cdots \partial x_N} T_H \equiv \delta.$$

Therefore, the derivative of the Heaviside function $H: \mathbb{R}^N \to \mathbb{R}$ is the Dirac delta distribution centered at the origin.

The next example should be compared with the Cauchy principal value distribution $\mathrm{vp}\,\frac{1}{x}$ described in **Section 8.4.1**.

**Example 9.5.** (DERIVATIVE OF $|x|^{-\alpha}$, $-\infty < \alpha < 1$) In $\mathbb{R}$ the real-valued function $u_\alpha: x \mapsto |x|^{-\alpha}$ with $-\infty < \alpha < 1$ is in $L^1_{\mathrm{loc}}(\mathbb{R})$ and therefore it is in $\mathcal{D}'(\mathbb{R})$. Let us consider its derivative in $\mathcal{D}'(\mathbb{R})$, that is the linear and continuous functional on $\mathcal{D}(\mathbb{R})$ defined by

$$u'_\alpha: \varphi \in \mathcal{D}(\mathbb{R}) \mapsto -\int_{\mathbb{R}} |x|^{-\alpha}\, \partial_x \varphi(x)\, \mathrm{d}x. \tag{9.2}$$

We want to find another representation of $u'_\alpha$. It is convenient to set, for any $r \in \mathbb{R}_+$, $K_r := [-r, r]$. We consider a generic $\varphi \in \mathcal{D}(\mathbb{R})$ and denote by $K_a := [-a, a]$ a compact subset of $\mathbb{R}$ such that $K_a \supseteq \mathrm{supp}_{\mathbb{R}}\, \varphi$. Clearly, for $\varepsilon$ sufficiently small (precisely when $0 < \varepsilon < a$) $K_a \backslash K_\varepsilon \neq \emptyset$ and we have (as $u_\alpha \in L^1_{\mathrm{loc}}(\mathbb{R})$)

$$\langle u'_\alpha, \varphi \rangle = -\lim_{\varepsilon \to 0^+} \int_{K_a \backslash K_\varepsilon} |x|^{-\alpha}\, \partial_x \varphi(x)\, \mathrm{d}x. \tag{9.3}$$

On the other hand, taking into accoung that $\varphi(a) = \varphi(-a) = 0$ because $K_a \supseteq \mathrm{supp}_{\mathbb{R}}\, \varphi$, for any $0 < \varepsilon < a$ we have (note that $K_a \backslash K_\varepsilon = [-a, -\varepsilon] \cup [\varepsilon, a]$)

$$\begin{aligned} \int_{K_a \backslash K_\varepsilon} |x|^{-\alpha}\, \partial_x \varphi(x)\, \mathrm{d}x &= \int_{K_a \backslash K_\varepsilon} \partial_x(|x|^{-\alpha} \varphi(x))\, \mathrm{d}x - \int_{K_a \backslash K_\varepsilon} \partial_x(|x|^{-\alpha}) \varphi(x)\, \mathrm{d}x \\ &= |\varepsilon|^{-\alpha}\, (\varphi(-\varepsilon) - \varphi(\varepsilon)) - \int_{K_a \backslash K_\varepsilon} \partial_x(|x|^{-\alpha}) \varphi(x)\, \mathrm{d}x. \end{aligned} \tag{9.4}$$

Summarizing, for any $0 < \varepsilon < a$ we have

$$\int_{K_a \backslash K_\varepsilon} \partial_x(|x|^{-\alpha}) \varphi(x)\, \mathrm{d}x = |\varepsilon|^{-\alpha}\, (\varphi(-\varepsilon) - \varphi(\varepsilon)) - \int_{K_a \backslash K_\varepsilon} |x|^{-\alpha}\, \partial_x \varphi(x)\, \mathrm{d}x. \tag{9.5}$$

Since $\varphi$ is differentiable, we have $\lim_{\varepsilon \to 0} |\varepsilon|^{-\alpha}\, (\varphi(-\varepsilon) - \varphi(\varepsilon)) = 0$ for any $-\infty < \alpha < 1$. Therefore, passing to the limit for $\varepsilon \to 0^+$ in (9.5) we infer that

$$\begin{aligned} \langle u'_\alpha, \varphi \rangle &= \lim_{\varepsilon \to 0^+} \int_{K_a \backslash K_\varepsilon} \partial_x(|x|^{-\alpha}) \varphi(x)\, \mathrm{d}x \\ &= \lim_{\varepsilon \to 0^+} \int_{K_a \backslash K_\varepsilon} -\alpha \frac{x}{|x|^{2+\alpha}} \varphi(x)\, \mathrm{d}x \\ &= -\alpha \left\langle \mathrm{vp}\left(\frac{x}{|x|^{2+\alpha}}\right), \varphi \right\rangle. \end{aligned} \tag{9.6}$$

By the arbitrariness of $\varphi \in \mathcal{D}(\mathbb{R})$ we get

$$u_\alpha' = -\alpha\,\mathrm{vp}\left(\frac{x}{|x|^{2+\alpha}}\right). \tag{9.7}$$

The previous distribution is an extension of the regular distribution in $\mathcal{D}'(\mathbb{R}\setminus\{0\})$ associated with the classical derivative of $|x|^{-\alpha}$. ⋯